MS-E1652 Computational methods for differential equations

(Final) Exam; 13:00–16:00, December 20, 2017 **Hyvönen**

1. Consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = 1, \tag{1}$$

for $t \geq 0$.

(a) Let $f(t,x) = x\cos(t)$ and prove that this function satisfies the (global) Lipschitz condition with respect to its second variable, i.e., that $|f(t,x) - f(t,y)| \leq L|x-y|$ for all $t, x, y \in \mathbb{R}$ and some $L > 0$. Give the unique solution of (1) explicitly.

(b) Let $f(t,x) = x^2\cos(t)$ and once again solve (1) explicitly. Is the solution well defined for all $t \in [0,\infty)$? Does this f satisfy the (global) Lipschitz condition with respect to its second variable?

2. Consider the *linear multistep method* (LMM)

$$x_{j+2} - \frac{1}{2}x_{j+1} - \frac{1}{2}x_j = \frac{3}{2}hf_{j+2}, \qquad j = 0, 1, 2, \dots . \tag{2}$$

(a) Is (2) an explicit or an implicit LMM?

(b) Prove that (2) is (at least) consistent of order $p = 1$.

(c) Is (2) zero-stable?

(d) Does the point $\hat{h} = -10$ belong to the region of absolute stability for (2)?

Justify your answers.

3. Consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0 \neq 0. \tag{3}$$

and the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{1}{4}h(k_1 + 3k_2), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + \frac{2}{3}h, x_j + \frac{2}{3}hk_1\Big). \end{aligned} \tag{4}$$

(a) Is (4) an explicit or an implicit RK method? Why?

(b) Let $f(t,x) = \lambda x$ with $\lambda \in \mathbb{C}$. Write x_j with the help of x_0 for an arbitrary $j \in \mathbb{N}$.

(c) Let $f(t,x) = \lambda x$ with $\lambda \in \mathbb{C}$. Prove that

$$x_1 = x(h) + O(h^3)$$

where $x : \mathbb{R}_+ \to \mathbb{C}$ is the exact solution of (3) for the considered f. Based on this result, what seems to be the order of the RK method (4)?

(d) Let $f(t,x) = \lambda x$ with a fixed *real* $\mathbb{R} \ni \lambda < 0$. For which values of the time step $h > 0$ does the numerical solution produced by (4) satisfy $x_j \to 0$ as $j \to \infty$?

4. When the initial/boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,1), \end{cases}$$

is spatially discretized by the standard central second order difference approximation, one ends up at the following initial value problem:

$$U'(t) = AU(t), \qquad U(0) = G, \tag{5}$$

for all $t \geq 0$. Here, $G = [g(x_1), \ldots, g(x_m)]^{\mathrm{T}}$ and $U(t) \approx [u(x_1,t), \ldots, u(x_m,t)]^{\mathrm{T}}$, with $x_j = jh$ and $h = 1/(m+1)$ being the mesh parameter.

(a) What does the difference matrix $A \in \mathbb{R}^{m \times m}$ look like? (It is enough to remember/reason the structure of A — you need not present an actual proof.)

(b) Introduce some consistent numerical method for solving (5). Let $\delta > 0$ be the time step size and denote by U_k the approximation of $U(k\delta)$ for $k = 0, 1, 2, \ldots$.

(c) For which $\delta > 0$ is your method (for sure) stable, that is,

$$\lim_{k \to \infty} U_k = 0 \in \mathbb{R}^m$$

for any $G \in \mathbb{R}^m$? Justify your answer.

Hint 1: The eigenvalues of $A \in \mathbb{R}^{m \times m}$ satisfy $0 > \lambda_1 > \lambda_2 > \cdots > \lambda_{m-1} > \lambda_m > -4/h^2$.

Hint 2: For any matrix $B \in \mathbb{R}^{m \times m}$, it holds that

$$\lim_{k \to \infty} B^k = 0 \in \mathbb{R}^{m \times m},$$

if and only if the eigenvalues of B satisfy $|\mu_j| < 1$, $j = 1, \ldots, m$.