MS-E1652 Computational methods for differential equations

Exam, 16:30–19:30, October 24, 2018 **Hyvönen/Candiani**

Calculators or other extra material are not allowed.

1. Consider the *linear multistep method* (LMM)

$$3x_{j+2} - 2x_{j+1} - x_j = 4hf_{j+2}, \qquad j = 0, 1, 2, \dots . \tag{1}$$

(a) Is (1) an explicit or an implicit LMM?

(b) Is (1) consistent (of any order $p > 0$)?

(c) Is (1) zero-stable?

(d) Does the point $\hat{h} = -5/4$ belong to the region of absolute stabilty for (1)?

Justify your answers.

2. Consider the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{h}{4}(3k_1 + k_2), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + h, x_j + hk_1\Big). \end{aligned} \tag{2}$$

(a) Is (2) an explicit or an implicit RK method? Why?

(b) What is the stability function corresponding to (2)?

(c) What is the region of absolute stability for (2)? (Hint: It is a disk.)

(d) Prove that (2) is *not* consistent of order $p = 2$. (Hint: Consider the stability function.)

3. Consider the initial value problem

$$u'(t) = \begin{bmatrix} -5 & -3 \\ 1 & -1 \end{bmatrix} u(t), \qquad u(0) = u_0. \tag{3}$$

(a) Prove that

$$\lim_{t \to \infty} u(t) = 0 \in \mathbb{R}^2$$

for any $u_0 \in \mathbb{R}^2$.

(b) For what values of the time step $h > 0$ does the numerical solution produced by the RK method (2) satisfy

$$\lim_{j \to \infty} u_j = 0 \in \mathbb{R}^2$$

for any $u_0 \in \mathbb{R}^2$. (If you were not able to deduce the answer to 2(c), explain how you would approach this problem setting if you knew the region of absolute stability for (2).)

4. When the initial/boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,1), \end{cases}$$

is spatially discretized by the standard central second order difference approximation, one ends up at the following initial value problem:

$$U'(t) = AU(t), \qquad U(0) = G, \tag{4}$$

for all $t \geq 0$. Here, $G = [g(x_1), \dots, g(x_m)]^{\mathrm{T}}$ and $U(t) \approx [u(x_1,t), \dots, u(x_m,t)]^{\mathrm{T}}$, with $x_j = jh$ and $h = 1/(m+1)$ being the mesh parameter.

(a) What does the difference matrix $A \in \mathbb{R}^{m\times m}$ look like? (It is enough to remember/reason the structure of A — you need not present an actual proof.)

(b) Introduce some consistent numerical method for solving (4). Let $\delta > 0$ be the time step size and denote by U_k the approximation of $U(k\delta)$ for $k = 0, 1, 2, \dots$.

(c) For which $\delta > 0$ is your method (for sure) stable, that is,

$$\lim_{k\to\infty} U_k = 0 \in \mathbb{R}^m$$

for any $G \in \mathbb{R}^m$? Justify your answer.

Hint 1: The eigenvalues of $A \in \mathbb{R}^{m\times m}$ satisfy $0 > \lambda_1 > \lambda_2 > \dots > \lambda_{m-1} > \lambda_m > -4/h^2$.

Hint 2: For any matrix $B \in \mathbb{R}^{m\times m}$, it holds that

$$\lim_{k\to\infty} B^k = 0 \in \mathbb{R}^{m\times m},$$

if and only if the eigenvalues of B satisfy $|\mu_j| < 1$, $j = 1, \dots, m$.