Aalto University

Period 3 Examination 2018

MS-C1080: Introduction to Abstract Algebra

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Time Allowed: 3 hours

Instructions for Candidates

The marks indicated in round brackets () at the end of each question will be awarded for complete and correct answers. The eighth question is a bonus question and will be awarded on top. It can be used to show excellence or for compensation.

Instructions for Invigilators

This examination is closed book. No written notes or other aiding material may be used during this examination.

Question 1: Write down a Cayley table of the group $\mathbb{Z}_8^\times$, and determine, whether or not this group is a cyclic group. (10)

Question 2: Let $\mathrm{GL}_2(\mathbb{R})$ denote the group of all invertible 2×2-matrices with real entries. Let

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) Show that A and B are indeed invertible, and determine their order. (5)

(b) Compute the product AB and determine its order. What is your conclusion? (5)

Question 3: Let G be a group, and let H and K be subgroups of G.

(a) Show that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$. (10)

(b) Show that $HK := \{hk \mid h \in H,\ k \in K\}$ is a subgroup of G if and only if $HK = KH$. (10)

Question 4: Show that the set $L(G)$ of all normal subgroups of a group G satisfies the so-called *Dedekind identity*, which means for all $A, B, C \trianglelefteq G$ with $A \leq C$, there holds (10)

$$(AB) \cap C = A(B \cap C).$$

Question 5: Let G be a group, and let A, B be subgroups of G, such that A is a normal subgroup. Assume further that $G = AB$ (which is the same as BA), and $A \cap B = \{e\}$. Show the following:

(a) For every $g \in G$ there exist unique $a, a' \in A$ and $b, b' \in B$ such that $g = ab = b'a'$. (5)

(b) There exists a homomorphism $\nu : G \longrightarrow B$ that fixes B elementwise such that $\ker(\nu) = A$. (5)

(c) For each $b \in B$ the mapping $\varphi_b : A \longrightarrow A,\ a \mapsto bab^{-1}$ is an automorphism of A, and the mapping $\varphi : B \longrightarrow \mathrm{Aut}(A),\ b \mapsto \varphi_b$ is a homomorphism. (5)

(d) $G \cong A \rtimes_\varphi B$, where $\rtimes_\varphi$ denotes the (outer) semi-direct product that we studied in the tutorial sessions. (5)

Question 6: An ideal I of a commutative ring R is said to be a *prime ideal* if $ab \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in R$.

(a) Show that an ideal of R is prime if and only if R/I is an integral domain. (10)

(b) What are the ideals in $\mathbb{Z}$? Which of these are prime ideals? (Proofs!) (10)

Question 7: Let R be a commutative ring. We take it as proven, that the binomial theorem holds, which means that for arbitrary $a, b \in R$ and $n \in \mathbb{N}$ there holds

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

The expressions $\binom{n}{i}$ have to be read as $\binom{n}{i}$ -fold sums of the identity 1, so that these are now elements of R. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some positive integer n. Show the following:

(a) If $a \in R$ is nilpotent, then $1 + a$ is an invertible element. (6)
Hint: Recall a well-known expression for $\frac{1}{1-x}$ and adapt it to this situation.

(b) The set $N(R)$ of all nilpotent elements in R forms a (two-sided) ideal of R. (6)

(c) The only nilpotent element of $R/N(R)$ is its zero element $N(R)$. (8)

Bonus Question 8: Let R and S be unital rings, and let $\varphi : R \longrightarrow S$ be a unital ring homomorphisms. Define $\ker(\varphi) := \{x \in R \mid \varphi(x) = 0\}$ and $\operatorname{im}(\varphi) := \{\varphi(x) \mid x \in R\}$.

(a) Show that $\ker(\varphi)$ is a two-sided ideal in R and $\operatorname{im}(\varphi)$ is a subring of S. (5)

(b) Show that φ is injective if and only if $\ker(\varphi) = \{0\}$. (5)

(c) Including proofs, derive the *first isomorphism theorem* saying that (10)

$$R/\ker(\varphi) \cong \operatorname{im}(\varphi).$$

Nota Bene: Questions 1 to 7 sum to 110 points, where 50 points are required to pass. Scores beyond 100 are considered excellent. I wish best success, and good luck!