Advanced Quantum Mechanics PHYS-E0414
Exam
13 December 2017

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Q1. (0.5p.) The Hamiltonian of a free spinless particle is $\hat{H} = -\frac{\hbar^2}{2m}\hat{\nabla}^2$. How is it modified in the presence of a time-independent magnetic field?

Q2. (0.5p.) The Hermitian operator $\hat{A}$ fulfills $\langle\phi|\hat{A}\psi\rangle = \langle\hat{A}\phi|\psi\rangle$ for any two wave functions $|\phi\rangle$ and $|\psi\rangle$. Can the eigenvalues of $\hat{A}$ be complex numbers? Demonstrate your answer.

Q3. (1p.) Let $\hat{H}$ be a Hermitian operator, with eigenvalues $\{E_n\}_{n=0}^{\infty}$, and eigenstates $\{|\psi_n\rangle\}_{n=0}^{\infty}$, so that $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$. Show that the average of $\hat{H}$ on the state $|\phi\rangle$ can be written as $\langle\hat{H}\rangle_\phi = \sum_m P_m E_m$. What is the meaning of P_m?

Q4. (1p.) In class we saw that unitarity and tensor product structure imply the no-cloning theorem. Here you will show that linearity and tensor product structures also imply the no-cloning theorem. Suppose a cloning unitary U exists for all inputs $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. Alice claims that

$$U|\psi\rangle \otimes |\text{blank}\rangle = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle .$$

However, Bob claims that:

$$U|\psi\rangle \otimes |\text{blank}\rangle = \alpha^2|0\rangle \otimes |0\rangle + \alpha\beta|0\rangle \otimes |1\rangle + \alpha\beta|1\rangle \otimes |0\rangle + \beta^2|1\rangle \otimes |1\rangle .$$

This contradiction can be used to show the no-cloning theorem.

(a) Elaborate in detail the steps that Alice and Bob each have in mind to reach these two equations.

(b) Under what condition on α and β are the two equations equivalent? What does this mean with respect to cloning?

Q5. (1.5p.) A density matrix (also sometimes known as a density operator) is a representation of statistical mixtures of quantum states. This exercise introduces some examples of density matrices, and explores some of their properties.

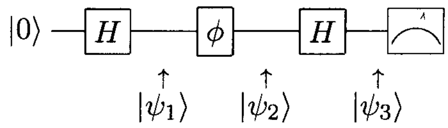
(a) Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ be a qubit state. Obtain the matrix $\rho = |\psi\rangle\langle\psi|$, which you may compute using linear algebra using the vector representations of $|\psi\rangle$ and $\langle\psi|$. What are the eigenvectors and eigenvalues of ρ in the $\{|0\rangle, |1\rangle\}$ basis?

(b) Let $\rho_0 = |0\rangle\langle 0|$ and $\rho_1 = |1\rangle\langle 1|$. Find the matrix $\sigma = \frac{\rho_0+\rho_1}{2}$. What are the eigenvalues and eigenvectors of σ?

(c) Compute $\mathrm{Tr}(\rho^2)$ and $\mathrm{Tr}(\sigma^2)$. Interpret the results.

Q6. (1.5p.) Consider the single qubit model of an interferometer, where the goal is to estimate an unknown phase ϕ (see figure). The ϕ operator maps $|0\rangle \to |0\rangle$ and $|1\rangle \to e^{i\phi}|1\rangle$.

(a) Give the states $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$.



(b) What is the probability of measuring the final qubit in state $|0\rangle$? And the probability to measure it in state $|1\rangle$? Plot and comment the results.

Hint: The Hadamard gate has the matrix representation $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Q7. (2p.) A spinless particle is confined in a one-dimensional infinite potential well of length a,

$$V(x) = \begin{cases} 0 & ,0 \leq x \leq a \\ \infty & ,\text{otherwise} \end{cases} .$$

The eigenenergies and eigenstates are

$$E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2 \quad , \quad \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) ,$$

with n an integer.

(a) Assuming $\lambda \ll 1$, compute the first-order energy shift to the levels with $n = 1, 2, 3$, given by the perturbation

$$V'(x) = \begin{cases} \frac{\lambda x}{a} & ,0 \leq x \leq a \\ 0 & ,\text{otherwise} \end{cases} .$$

Hint:

$$\int \mathrm{d}x \sin^2(\alpha x) x = \frac{x^2}{4} - \frac{\cos(2\alpha x)}{8\alpha^2} - x\frac{\sin(2\alpha x)}{4\alpha}$$

(b) Consider now that there are two *identical fermions* in the potential well (without perturbation). What is the energy and wave function of the ground state (ignoring the spin degree of freedom)?

Q8. (2p.) The Hamiltonian of two interacting spin-1/2 particles is given by $\hat{H} = \gamma \hat{S}_{(1)} \cdot \hat{S}_{(2)}$, where $\hat{S}_{(1)}$ and $\hat{S}_{(2)}$ are the spin operators for particles 1 and 2, respectively.

(a) Write the matrix representation of $\hat{H}$ in the basis of two spins, i.e., $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

(b) Find the eigenvalues and eigenvectors of $\hat{H}$.

Hint: You can use the following properties of the angular momentum operators:

$$\begin{aligned} J_i^2 |j_1, m_1; j_2, m_2\rangle &= \hbar^2 j_i(j_i + 1) |j_1, m_1; j_2, m_2\rangle , \\ J_{i,z} |j_1, m_1; j_2, m_2\rangle &= \hbar m_i |j_1, m_1; j_2, m_2\rangle , \\ J^2 |j_1, j_2; jm\rangle &= \hbar^2 j(j + 1) |j_1, j_2; jm\rangle , \\ J_z |j_1, j_2; jm\rangle &= \hbar m |j_1, j_2; jm\rangle , \end{aligned}$$

with $i = 1, 2$. Additionally, for the ladder operators $J_\pm = J_x \pm iJ_y$, we have

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j + 1) - m(m \pm 1)} |j, m \pm 1\rangle .$$