ELEC-E8101 Digital and Optimal Control

Final Exam (13.12.2017) - Solution

1. For each of the following statements, state if it is correct or not. Justify your answer.

   (a) For removing the steady-state error from, e.g., a step-response of a system, we need to apply derivative control. [1p]

   (b) The eigenvalues to a general system $\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k]$ can always be placed anywhere using state-feedback. [1p]

   (c) If a linear system is reachable the state can be made to follow any trajectory. [1p]

   (d) If linear system is reachable the state can be made to go between any two initial and final values. [1p]

   (e) Suppose $x(t)$ is a low-pass signal with $X(i\omega) = 0 \ \forall|\omega| > |\omega_0|$. Then, $x(t)$ can be uniquely determined if its samples $x(nT_s), n = 0, \pm1, \pm2, \ldots$ satisfy $\dfrac{2\pi}{T_s} > 2\omega_0$. [1p]

   (f) When discretizing a continuous-time system with poles $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $|\lambda_{\max}| \triangleq \max_i |\lambda_i|$, using the state-space representation with sampling h and zero-order hold (ZOH), then the stability of the analog system is preserved and if $h < \pi/|\lambda_{\max}|$ there is no aliasing. [2p]

   (g) The separation principle tells us that the control design and the observer design cannot be done independently of each other. [1p]

   (h) Given the sequences $\{x[k_1], x[k_2], \ldots, x[k_n]\}$ and $\{x[k_1+k], x[k_2+k], \ldots, x[k_n+k]\}$. We say that the stochastic process is called *stationary* if the two first moments of the distributions (mean and covariance) are the same for all values of k. [1p]

   (i) The optimal predictor is better when considering a large prediction horizon. [1p]

   Solution.

   (a) **False**. The steady-state error is removed by applying integral control. Derivative control aims at reducing the changes in the state.

   (b) **False**. Arbitrary eigenvalue-placement is only possible when the system is reachable (completely controllable).

   (c) **False**. If a system is reachable, then it can move between any two states. But, it can not move (in general) along any trajectory.

   (d) **True**. As stated above, a reachable system can move between any two states.

   (e) **True**. Otherwise, there is aliasing.

   (f) **True**. This can be divided into 2 parts: stability and aliasing.
   Stability: Let $\lambda_i(A) = -\sigma_i + j\omega_i, \ \sigma_i > 0$. Then:

$$\lambda_i(\Phi) = e^{(-\sigma_i + j\omega_i)h} = e^{-\sigma_i h} e^{j\omega_i h} \Rightarrow |\lambda_i(\Phi)| = e^{-\sigma_i h}|e^{j\omega_i h}| = e^{-\sigma_i h} < 1$$

(Turn over)

Aliasing: To avoid aliasing:

$$\frac{2\pi}{h} > 2\omega_i \Rightarrow h < \frac{\pi}{|\lambda_{\max}|} \leq \frac{\pi}{\omega_i}$$

(g) **False**. Actually, it tells us the complete opposite – that control and observer design can be decoupled.

(h) **False**. A stochastic process is called stationary if the finite-dimensional distributions of $\{x[k_1], x[k_2], \ldots, x[k_n]\}$ and $\{x[k_1+k], x[k_2+k], \ldots, x[k_n+k]\}$ are identical. The process is *weakly stationary*, if the two first moments of the distributions (mean and covariance) are the same for all values of k. The value of the covariance function then depends only on the time difference.

(i) **False**. The optimal predictor is better when considering a small prediction horizon (cf. $\sigma_{\tilde{y}}^2 = (1 + f_1^2 + \ldots + f_{m-1}^2)\sigma_e^2$).

2. a) Consider the following ARMA-process:

$$y[k+2] - 1.5y[k+1] + 0.5y[k] = 2\big(e[k+2] - 1.2e[k+1] + 0.6e[k]\big),$$

where $\{e[k]\}$ is a sequence of independent normal random variables with zero mean and unit variance. Determine the 2-step predictor which minimizes the mean square prediction error. [5p]

b) Consider the following ARMAX-process:

$$y[k+3] - 1.7y[k+2] + 0.7y[k+1] = u[k+1] + 0.5u[k] + e[k+3] - 0.9e[k+2],$$

where $\{e[k]\}$ is a sequence of independent normal random variables with zero mean and unit variance. Determine the minimum-variance controller and the minimum achievable variance. [5p]

Solution.

a) The pulse transfer function (using the shift operator) is:

$$y[k] = \frac{C'(q)}{A(q)}e[k] = \frac{2(q^2 - 1.2q + 0.6)}{q^2 - 1.5q + 0.5}e[k]$$

The polynomials $C'(q)$ and $A(q)$ have to be *monic*, i.e., the coefficient of the highest q-term of the polynomial has to be one. Only then the equations for the m-step predictor hold. Obviously, $C'(q)$ is not monic, thus we have to make it monic by doing the following: let $v[k] \triangleq 2e[k]$. Then,

$$C'(q)e[k] = \underbrace{(q^2 - 1.2q + 0.6)}_{C(q)}v[k]$$

Note that

$$E\{v[k]\} = E\{2e[k]\} = 2E\{e[k]\} = 0$$
$$\text{var}\{v[k]\} \triangleq \sigma_v^2 = E\{v^2[k]\} = E\{4e^2[k]\} = 4E\{e^2[k]\} = 4$$

Now the process model is:

$$y[k] = \frac{C(q)}{A(q)}v[k] = \frac{q^2 - 1.2q + 0.6}{q^2 - 1.5q + 0.5}v[k]$$

Now, we need to solve the equation

$$q^{m-1}C(q) = A(q)F(q) + G(q),$$

for $F(q)$ and $G(q)$, where

$$F(q) = q^{m-1} + f_1q^{m-2} + \dots f_{m-1},$$
$$G(q) = g_0q^{n-1} + g_1q^{n-2} + \dots g_{n-1}.$$

Since we are dealing with a 2-step predictor (m=2), $F(q)$ and $G(q)$ become

$$F(q) = q + f_1$$
$$G(q) = g_0q + g_1$$

(Turn over)

Therefore,

$$
\begin{aligned}
q(q^2 - 1.2q + 0.6) &\equiv (q^2 - 1.5q + 0.5)(q + f_1) + (g_0 q + g_1) \\
q^3 - 1.2q^2 + 0.6q &\equiv q^3 + (f_1 - 1.5)q^2 + (0.5 + g_0 - 1.5f_1)q + (0.5f_1 + g_1) \\
&\Rightarrow \begin{cases} f_1 = 0.3 \\ g_0 = 0.6 - 0.5 + 1.5(0.3) = 0.1 + 0.45 = 0.55 \\ g_1 = -0.5(0.3) = -0.15 \end{cases}
\end{aligned}
$$

Therefore, the 2-step predictor is given by

$$
\hat{y}[k+2|k] = \frac{qG(q)}{C(q)} y[k] = \frac{0.55q^2 - 0.15q}{q^2 - 1.2q + 0.6} y[k]
$$

and the variance of the prediction error:

$$
\text{var}\{y[k+1|k]\} = (1 + f_1^2)\sigma_v^2 = \left(1 + (0.3)^2\right) 4 = (1.09)4 = 4.36.
$$

b) The pulse transfer function (using the shift operator) is:

$$
\begin{aligned}
(q^3 - 1.7q^2 + 0.7q)y[k] &= (q + 0.5)u[k] + (q^3 - 0.9q^2)e[k] \\
y[k] &= \frac{q + 0.5}{q^3 - 1.7q^2 + 0.7q} u[k] + \frac{q^3 - 0.9q^2}{q^3 - 1.7q^2 + 0.7q} e[k]
\end{aligned}
$$

Hence, we get the polynomials:

$$
\begin{cases} A(q) &= q^3 - 1.7q^2 + 0.7q \\ B(q) &= q + 0.5 \\ C(q) &= q^3 - 0.9q^2 \end{cases}
$$

Now, we need to solve the equation

$$
q^{d-1}C(q) = F(q)A(q) + G(q).
$$

for $F(q)$ and $G(q)$, where

$$
\begin{aligned}
F(q) &= q^{d-1} + f_1 q^{d-2} + \ldots + f_{d-1} \\
G(q) &= g_0 q^{n-1} + g_1 q^{n-2} + \ldots + g_{n-1}
\end{aligned}
$$

where d is pole excess of the system

$$
d = \text{deg}A - \text{deg}B.
$$

Therefore, since $d = 3 - 1 = 2$, we get

$$
\begin{aligned}
q(q^3 - 0.9q^2) &\equiv (q^3 - 1.7q^2 + 0.7q)(q + f_1) + (g_0 q^2 + g_1 q + g_2) \\
q^4 - 0.9q^3 &\equiv q^4 + (f_1 - 1.7)q^3 + (g_0 - 1.7f_1 + 0.7)q^2 + (0.7f_1 + g_1)q + g_2
\end{aligned}
$$

$$
\Rightarrow \begin{cases} f_1 - 1.7 = -0.9 \Rightarrow f_1 = 0.8 \\ g_0 = 1.7f_1 - 0.7 = 1.7(0.8) - 0.7 = 1.36 - 0.7 = 0.66 \\ g_1 = -0.7f_1 = -0.7(0.8) = -0.56 \\ g_2 = 0 \end{cases}
$$

This gives the polynomials

$$F(q) = q + 0.8$$
$$G(q) = 0.66q^2 - 0.56q$$

and the minimum variance control law becomes:

$$u[k] = -\frac{G(q)}{B(q)F(q)}y[k] = -\frac{0.66q^2 - 0.56q}{(q+0.5)(q+0.8)}y[k].$$

The variance of the output y is:

$$\text{var}\{y[k]\} = (1 + f_1^2)\sigma_e^2 = \left(1 + (0.8)^2\right) 1 = 1.64.$$

(Turn over)

3. The double integrator is a common process in mechanical models. Its differential equation form is

$$\frac{d^2y(t)}{dt^2} = u(t).$$

a) Show that the *discrete-time* state-space representation of the double integrator when we use ZOH and sampling time h is given by [5p]

$$\begin{aligned}\mathbf{x}(kh+h) &= \begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}\mathbf{x}(kh) + \begin{bmatrix}h^2/2\\ h\end{bmatrix}u(kh)\\ y(kh) &= \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(kh)\end{aligned}$$

Hint: Find the continuous-time state-space representation first.

b) We consider linear, static state-feedback, i.e., $u[k] = -L\mathbf{x}[k] = -\begin{bmatrix}l_1 & l_2\end{bmatrix}\mathbf{x}[k]$ for controlling the sampled double-integrator plant. The desired closed-loop poles are given by the following characteristic equation:

$$z^2 + p_1 z + p_2 = 0.$$

Show that the controller parameter vector $L = \begin{bmatrix}l_1 & l_2\end{bmatrix}$ is given by [4p]

$$\begin{aligned}l_1 &= \frac{1}{h^2}(1 + p_1 + p_2),\\ l_2 &= \frac{1}{2h}(3 + p_1 - p_2).\end{aligned}$$

c) Determine the deadbeat controller L for $h = 1$. [2p]

d) Design a deadbeat-type state observer to the system. [4p]

Solution.

a) Set $x_1 = y$, $x_2 = dy/dt$ and $\mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$. Then,

$$\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{d^2y}{dt^2} = u(t)\end{aligned}\right\} \Rightarrow \begin{cases}\dot{\mathbf{x}}(t) &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\mathbf{x}(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(t)\end{cases}$$

We need to find $\Phi(h)$ and $\Gamma(h)$:

$$\begin{aligned}\Phi(h) &= e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}h^nA^n\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + h\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \mathbf{0} = \begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}\\ \Gamma(h) &= \int_0^h e^{As}ds B = \int_0^h \begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix} ds \begin{bmatrix}0\\ 1\end{bmatrix} = \int_0^h \begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} ds\\ &= \int_0^h \begin{bmatrix}s\\ 1\end{bmatrix} ds = \begin{bmatrix}h^2/2\\ h\end{bmatrix}\end{aligned}$$

Therefore,

$$\mathbf{x}(kh+h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \mathbf{u}(kh)$$
$$\mathbf{y}(kh) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kh)$$

b) With the controller $u[k] = -L\mathbf{x}[k] = -\begin{bmatrix} l_1 & l_2 \end{bmatrix} \mathbf{x}[k]$, the system becomes:

$$\mathbf{x}[k+1] = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}[k] - \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} \mathbf{x}[k] = \begin{bmatrix} 1 - l_1 h^2/2 & h - l_2 h^2/2 \\ -l_1 h & 1 - l_2 h \end{bmatrix} \mathbf{x}[k]$$

Take the characteristic equation:

$$z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2 \right) z - \left(\frac{l_1 h^2}{2} - l_2 h + 1 \right) = 0$$

Now, we compare with the desired characteristic equation

$$\Rightarrow \begin{cases} \frac{l_1 h^2}{2} + l_2 h - 2 = p_1 \\ \frac{l_1 h^2}{2} - l_2 h + 1 = p_2 \end{cases} \Rightarrow \begin{cases} l_1 = \frac{1}{h^2}(1 + p_1 + p_2) \\ l_2 = \frac{1}{2h}(3 + p_1 - p_2) \end{cases}$$

c) For deadbeat control $p_1 = p_2 = 0$. Therefore, for $h = 1$,

$$\begin{cases} l_1 = \frac{1}{h^2}(1 + p_1 + p_2) \\ l_2 = \frac{1}{2h}(3 + p_1 - p_2) \end{cases} \Rightarrow \begin{cases} l_1 = 1 \\ l_2 = \frac{3}{2} \end{cases}$$

d) The characteristic equation is given by

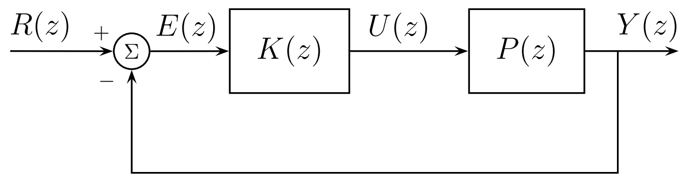
$$\begin{aligned} \det(zI - \Phi_o) &= \det(zI - \Phi + KC) \\ &= \det\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \\ &= \begin{vmatrix} z - 1 + k_1 & -h \\ k_2 & z - 1 \end{vmatrix} \\ &= (z - 1 + k_1)(z - 1) + k_2 h \\ &= z^2 - z + k_1 z - z + 1 - k_1 + k_2 h \\ &= z^2 + (k_1 - 2)z + (k_2 h - k_1 + 1) = 0 \end{aligned}$$

Therefore, for having the poles at zero,

$$\begin{cases} k_1 = 2 \\ k_2 = \frac{k_1 - 1}{h} = \frac{1}{h}. \end{cases}$$

(Turn over)

4. Consider the system described by the figure below:



where input signal $r[k]$ is the input and the pulse transfer function of the process is

$$P(z) = \frac{1}{z - 0.5}.$$

a) Suppose $K(z) = K, K > 0$ (Proportional-controller). The control algorithm is

$$u(kh) = K\big(r(kh - \tau) - y(kh - \tau)\big),$$

where τ is a computational/processing delay.

i) Determine the values of gain K for which the closed-loop system is stable, for (1) $\tau = 0$ and (2) $\tau = h$. [5p]

ii) Consider the closed-loop system and let the input $r[k]$ be a unit step. Find, as a function of gain K, the steady-state value of $y[k]$ (i.e., the $\lim_{k\to\infty} y[k]$) when this is finite, stating for which values of K the answer is valid, for $\tau = 0$ and $\tau = h$. [4p]

iii) Is there a gain K for which you can achieve a zero steady-state error? [1p]

b) You want the steady state error $e[k]$ (i.e., $\lim_{k\to\infty} e[k]$) to be zero when the input $r[k]$ is a unit step. Design a controller $K(z)$ (or modify the existing one) such that the system (1) is stable and (2) has zero steady-state error for $\tau = h$. [5p]

Solution.

a) i) $\underline{\tau = 0}$: The closed-loop pulse transfer function is:

$$G_{cl}(z) = \frac{KP(z)}{1 + KP(z)} = \frac{K\frac{1}{z-0.5}}{1 + K\frac{1}{z-0.5}} = \frac{K}{z - 0.5 + K}.$$

This pulse transfer function has pole at $p_1 = 0.5 - K$. The stability condition is: $|p_1| < 1$. Therefore,

$$|0.5 - K| < 1 \Rightarrow -1 < 0.5 - K < 1 \Rightarrow -0.5 < K < 1.5.$$

Note that it was already assumed in the problem statement that $K > 0$, therefore $0 < K < 1.5$.

$\underline{\tau = h}$: The control law now becomes

$$u(kh) = K\big(r(kh - h) - y(kh - h)\big).$$

Taking z-transforms:

$$U(z) = K\big(z^{-1}R(z) - z^{-1}Y(z)\big) = Kz^{-1}(R(z) - Y(z)) = Kz^{-1}E(z).$$

Hence, the closed-loop pulse transfer function becomes:

$$G_{cl}(z) = \frac{K(z)P(z)}{1 + K(z)P(z)} = \frac{\frac{K}{z}\frac{1}{z-0.5}}{1 + \frac{K}{z}\frac{1}{z-0.5}} = \frac{K}{z(z-0.5)+K}.$$

The characteristic equation is, therefore, $z^2 - 0.5z + K = 0$. Taking the triangle rule:

$$\begin{cases} K < 1 \\ K > -(-0.5) - 1 = -0.5 \\ K > (-0.5) - 1 = -1.5 \end{cases}$$

Since it was already assumed in the problem statement that $K > 0$, therefore $0 < K < 1$.

ii) $\underline{\tau = 0}$: Applying the Final Value Theorem (FVT):

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1}(1-z)Y(z) = \lim_{z\to 1}\left\{\frac{K}{z-0.5+K}\right\} = \frac{K}{K+0.5}$$

The answer is valid for the values of K for which the system is stable, i.e., $0 < K < 1.5$.

$\underline{\tau = h}$: Applying the Final Value Theorem (FVT):

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1}(1-z)Y(z) = \lim_{z\to 1}\left\{\frac{K}{z(z-0.5)+K}\right\} = \frac{K}{K+0.5}$$

The answer is valid for the values of K for which the system is stable, i.e., $0 < K < 1$.

iii) No. As it can be seen by the FVT, only if $K \to \infty$ the steady-state error would go to zero, but the system becomes unstable much earlier.

b) There are many ways to solve this one. Here, I give a simple approach (for $\tau = 0$), in which we consider a PI-controller with equal gains. i.e.,

$$K(z) = K + \frac{K}{z-1} = \frac{K(z-1)+K}{z-1} = \frac{Kz}{z-1}$$

Hence, the closed-loop pulse transfer function becomes:

$$G_{cl}(z) = \frac{K(z)P(z)}{1+K(z)P(z)} = \frac{\frac{Kz}{z-1}\frac{1}{z-0.5}}{1+\frac{Kz}{z-1}\frac{1}{z-0.5}} = \frac{Kz}{(z-1)(z-0.5)+Kz} = \frac{Kz}{z^2+(K-1.5)z+0.5}.$$

The characteristic equation is, therefore, $z^2 + (K-1.5)z + 0.5 = 0$. Taking the triangle rule:

$$\begin{cases} 0.5 < 1 \\ 0.5 > -(K-1.5) - 1 \Rightarrow K > 1.5 - 1 - 0.5 = 0 \\ 0.5 > (K-1.5) - 1 \Rightarrow K < 3 \end{cases}$$

Therefore, $0 < K < 3$.

Applying the Final Value Theorem (FVT):

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1}(1-z)Y(z) = \lim_{z\to 1}\left\{\frac{Kz}{z^2+(K-1.5)z+0.5}\right\} = \frac{K}{1+K-1.5+0.5} = \frac{K}{K} = 1$$

When $\tau = h$, the closed-loop pulse transfer function becomes:

$$G_{cl}(z) = \frac{K(z)P(z)}{1+K(z)P(z)} = \frac{\frac{K}{z-1}\frac{1}{z-0.5}}{1+\frac{K}{z-1}\frac{1}{z-0.5}} = \frac{K}{(z-1)(z-0.5)+K} = \frac{K}{z^2-1.5z+(0.5+K)}.$$

The characteristic equation is, therefore, $z^2 - 1.5z + (0.5+K) = 0$. Taking the triangle rule:

$$\begin{cases} 0.5+K < 1 \Rightarrow K < 0.5 \\ 0.5+K > -(-1.5)-1 \Rightarrow K > 1.5-1-0.5 = 0 \\ 0.5+K > (-1.5)-1 \Rightarrow K > -3 \end{cases}$$

Therefore, $0 < K < 0.5$.

Applying the Final Value Theorem (FVT):

$$\lim_{k\to\infty} y[k] = \lim_{z\to 1}(1-z)Y(z) = \lim_{z\to 1}\left\{\frac{K}{z^2-1.5z+(0.5+K)}\right\} = \frac{K}{1-1.5+0.5+K} = \frac{K}{K} = 1$$