MS-C2111 Stochastic processes **Exam**
Department of Mathematics and Systems Analysis 12.12.2018, 16:30–19:30
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You are allowed to use a **calculator** approved by the Finnish Matriculation Examination Board and a handwritten **cheat sheet**. The cheat sheet must be of size a4 with text only on one side, and it must contain your name and student number in the upper right corner. You don't need to return the cheat sheet. The exam contains 4 problems, each worth 0–6 points.

1. Two computer algorithms called muGo and nuGo compete in a go tournament where the winner is declared to be the one who first wins two consecutive games in a row. Due to more powerful computing hardware, muGo is expected to win each game with probability $p = 0.75$, independently of other games. The state of the tournament after t rounds is denoted by (M_t, N_t) where M_t equals the number of consecutive games won by muGo, and N_t is the corresponding number for nuGo.

(a) Model the state of the tournament as a Markov chain on a finite state space, draw its transition diagram, and write down its transition matrix. (2 p)

(b) What is the probability that muGo wins the tournament? (2 p)

(c) Does the Markov chain have an invariant distribution? If yes, how many? (2 p)

2. An imaginary bird species reproduces as follows. During its lifetime, every individual lays two eggs, and from each egg a new bird is born with probability 2/3 — independently of others. The first generation contains three birds.

(a) What is the probability that there are no birds in the third generation? (2 p)

(b) What is the probability that the bird species eventually becomes extinct? (4 p)

3. Let $(M_0, M_1, M_2, \dots)$ be a time-homogeneous discrete-time Markov chain on state space $S = \{1, 2, 3\}$ with transition matrix P and a random initial state M_0 which is uniformly distributed in S. A stranger on a street claims that (M_t) is a martingale with respect to its own information.

(a) Give an example of P for which the claim is true, and explain why. (2 p)

(b) Give an example of P for which the claim is not true, and explain why. (2 p)

(c) Write down a necessary and sufficient condition for P, under which the claim is true. (2 p)

4. Cars of type $i = 1, 2$ arrive to a one-way street at random time instants with expected interarrival times of ℓ_i minutes ($\ell_1 = 3$, $\ell_2 = 10$). The street has one parking space. If an arriving car finds the space vacant, the car parks there immediately. Otherwise the car drives away. A car of type i remains parked for an expected duration of m_i minutes ($m_1 = 5$, $m_2 = 20$). The interarrival times and parking times of type-i cars are mutually independent and exponentially distributed. In addition, type-1 cars behave independently of type-2 cars. Currently the parking space is vacant.

(a) Model the state of the parking space as a continuous-time Markov chain. Draw the transition diagram and write down the generator matrix of the chain. (2 p)

(b) Determine the probability that the parking space is vacant in steady state. (2 p)

(c) Explain how you can compute with the help of a computer the probability that the parking space is vacant after 30 minutes from now. (1 p)

(d) What is the probability that at least 2 cars arrive to the street during the 5-minute time interval starting after one hour? (1 p)