Aalto University School of Science
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MS-C1350 Partial differential equations, fall 2018

Course exam on 17 Dec 2018 at 9:00-12:00

This set of problems is for the participants of the course in the fall 2018 and affects 50% in the grading of the course.

1. What are the six functions listed below?

   (a) $u(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}$, $x \neq 0$, where $\alpha(n)$ the volume of the unit ball in $\mathbb{R}^n$,

   (b) $\widehat{u}(\xi, y) = \widehat{g}(\xi) e^{-|\xi| y}$, $y > 0$,

   (c) $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$, $t > 0$,

   (d) $u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y)$,

   (e) $u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|} \right) e^{ix\cdot\xi} \, d\xi$,

   (f) $u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} (th(y) + g(y) + \nabla g(y) \cdot (y - x)) \, dS(y)$.

   Plain answers without any explanations are enough. Please choose your answer from the following list:

   - the heat kernel in the upper half-space,
   - the solution on the Fourier side of the Cauchy problem for the wave equation,
   - the solution on the Fourier side of the Dirichlet problem for the Laplace equation in the upper half-space,
   - the fundamental solution of the Laplace equation when the dimension is at least three,
   - the solution of the Dirichlet problem for the Laplace equation in a ball,
   - Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation.

2. Assume that $u = u(x,t)$ is a solution of $u_{tt} - c^2\Delta u = 0$, with $c > 0$. Show that $v(x,t) = u(x, \frac{t}{c})$ is a solution of $v_{tt} - \Delta v = 0$.

3. How can the Dirichlet problem for the Laplace equation in the unit ball in the two-dimensional case be solved by separation of variables? A brief list of the main steps of the argument is enough.

4. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $T > 0$. Denote

$$\Omega_T = \Omega \times (0,T) \quad \text{and} \quad \Gamma_T = (\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,T]).$$

Consider a solution $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ to the heat equation in Ω_T.

(a) State the maximum principle for the heat equation in Ω_T. A precise statement of the maximum principle without any proofs is enough.

(b) State the initial–boundary value problem for the heat equation in Ω_T.

(c) Show that the solution of the initial–boundary value problem for the heat equation in Ω_T is unique by applying the maximum principle.

5. Give solution formulas to the problems below in terms of the fundamental solution of the heat equation and the initial data. It is enough to give the explicit formulas without any proofs.

(a) The Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^n)$.

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \quad t > 0, \\ 0, & x \in \mathbb{R}^n, \quad t \leq 0. \end{cases}$$

(b) The nonhomogeneous Cauchy problem with zero boundary values

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

where $f \in C_0^\infty(\mathbb{R}^n)$. Duhamel's principle asserts that

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

is a solution to this problem, where $u(x,t;s)$, with $0 < s < t$, is a solution the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, & x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

(c) The nonhomogeneous Cauchy problem with general boundary values

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $f, g \in C_0^\infty(\mathbb{R}^n)$.