Advanced Quantum Mechanics PHYS-E0414
12 December 2018

Exam

Q1. (1p.) Consider the spatial and temporal translation operators $\hat{\mathcal{T}}(\delta x)\,|x\rangle = |x+\delta x\rangle$ and $\hat{U}(\delta t)\psi(x,t) = \psi(x, t+\delta t)$. For infinitesimal δx and δt, $\hat{\mathcal{T}}(\delta x) = \hat{I} - \frac{i\delta x}{\hbar}\hat{p}$ and $\hat{U}(\delta t) = \hat{I} - \frac{i\delta t}{\hbar}\hat{H}$.

a) Explain why in the expressions above we must have the identity $\hat{I}$, the imaginary i, Planck's (reduced) constant $\hbar$, the momentum $\hat{p}$, and the Hamiltonian $\hat{H}$.

b) For time-independent $\hat{p}$ and $\hat{H}$, we have in general the finite translations $\hat{\mathcal{T}}(\Delta x) = \exp(-i\Delta x\hat{p}/\hbar)$ and $\hat{U}(\Delta t) = \exp(-i\Delta t\hat{H}/\hbar)$. Verify that they yield the correct infinitesimal expressions. For a free particle, show that

$$\psi(x+\Delta x, t+\Delta t) = \exp(-iE\Delta t/\hbar + iP\Delta x/\hbar)\psi(x,t),$$

with E and P the eigenvalues of the Hamiltonian and the momentum operator, respectively.

Q2. (1p.) Consider the following equation in position space

$$(\nabla^2 + k^2)G(\vec{r}) = \delta^3(\vec{r}).$$

a) How is it related to time-independent scattering problems?

b) Find the Fourier transform of the Green's function $G(\vec{r})$ satisfying the equation.

Q3. (0.5p) Formulate Bloch's theorem and very briefly explain how it is applied.

Q4. (1p.) The operator associated with the velocity of a particle with mass m can be defined as $\hat{\mathbf{V}} = \hat{\mathbf{P}}/m$, with $\hat{\mathbf{P}}$ the momentum operator.

(a) How is the velocity operator modified in the presence of a uniform magnetic field?

(b) Consider a uniform magnetic field in an arbitrary direction of space, calculate the commutator $[\hat{V}_x, \hat{V}_y]$.

Q5. (2p.) The Hamiltonian of two interacting spin-1/2 particles is given by $\hat{H}=\gamma\hat{\mathbf{S}}_{(1)}\cdot\hat{\mathbf{S}}_{(2)}$, where $\hat{\mathbf{S}}_{(1)}$ and $\hat{\mathbf{S}}_{(2)}$ are the spin operators for particles 1 and 2, respectively, and γ is a constant. Write the matrix representation of $\hat{H}$ in the basis of two spins, i.e., $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and find the eigenvalues of $\hat{H}$.

Hint: You can use the following properties of the angular momentum operators:

$$\begin{aligned}
\hat{J}_i^2|j_1, m_1; j_2, m_2\rangle &= \hbar^2 j_i(j_i+1)|j_1, m_1; j_2, m_2\rangle\,, \\
\hat{J}_{i,z}|j_1, m_1; j_2, m_2\rangle &= \hbar m_i|j_1, m_1; j_2, m_2\rangle\,, \\
\hat{J}^2|j_1, j_2; jm\rangle &= \hbar^2 j(j+1)|j_1, j_2; jm\rangle\,, \\
\hat{J}_z|j_1, j_2; jm\rangle &= \hbar m|j_1, j_2; jm\rangle\,,
\end{aligned}$$

with $i=1,2$. Additionally, for the ladder operators $\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y$, we have

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}|j, m\pm 1\rangle\,.$$

Q6. (1p.) The entropy of a quantum state, expressed as a density matrix ρ, is $S(\rho) = -\mathrm{Tr}\,(\rho\log_2\rho)$; in terms of its eigenvalues λ_k this is $S(\rho) = -\sum_k \lambda_k \log_2 \lambda_k$

(a) Calculate the entropy $S(\rho_0)$ for $\rho_0 = |0\rangle\langle 0|$.

(b) Calculate the entropy of $\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.

(c) A state ρ is a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$. Prove that this expression is equivalent to $S(\rho) = 0$ for positive eigenvalues of ρ.

(d) Find whether ρ_0 or ρ_1 is a pure state.

Q7. (2p.) Consider a one-dimensional infinite potential well of length L, with single-particle eigenenergies E_n and eigenstates ϕ_n, for integer n:

$$V(x) = \begin{cases} 0 & ,0\le x\le L \\ \infty & ,\text{otherwise} \end{cases}\quad,\quad E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2,\quad \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right).$$

(a) Suppose two *identical spin-zero bosons* are placed in the potential well. Find the wave functions and energies for the ground state and the first two excited states of the system.

(b) Consider now that the two bosons interact with each other through the perturbing potential $V'(x_1, x_2) = -V_0 L\delta(x_1 - x_2)$, with V_0 real and $\delta(x_1 - x_2)$ the Dirac delta function. Assume $V_0 \ll 1$ and compute the first-order correction to the ground state energy of the system.

Hint:

$$\int_0^\beta \mathrm{d}x\,\sin^4(\alpha x) = \frac{1}{32\alpha}\left[12\alpha\beta + \sin(4\alpha\beta) - 8\sin(2\alpha\beta)\right]$$

Q8. (1.5p.) The Fredkin gate (C_{SWAP}) is a circuit that transmits the first bit (control bit) unchanged and swaps the last two bits if, and only if, the first bit is one. For example:

$$C_{\text{SWAP}}\left(|0\rangle_c \otimes |1\rangle_t \otimes |0\rangle_t \right) = \left(|0\rangle_c \otimes |1\rangle_t \otimes |0\rangle_t \right)$$
$$C_{\text{SWAP}}\left(|1\rangle_c \otimes |1\rangle_t \otimes |0\rangle_t \right) = \left(|1\rangle_c \otimes |0\rangle_t \otimes |1\rangle_t \right)$$

The Toffoli gate (C_{CNOT}) is a circuit that inverts the target bit if the two control bits are both set to 1, otherwise all bits stay the same. For example:

$$C_{\text{CNOT}}\left(|1\rangle_c \otimes |0\rangle_t \otimes |1\rangle_c \right) = |1\rangle_c \otimes |1\rangle_t \otimes |1\rangle_c$$
$$C_{\text{CNOT}}\left(|0\rangle_c \otimes |1\rangle_c \otimes |0\rangle_t \right) = |0\rangle_c \otimes |1\rangle_c \otimes |0\rangle_t$$

Construct the Fredkin gate by using three Toffoli gates.

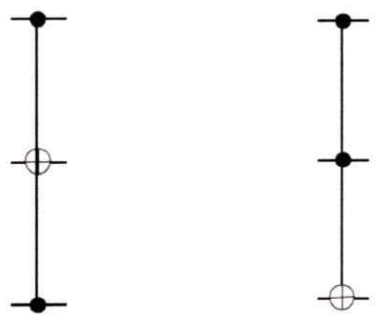

Figure: Examples of Two Toffoli gates with • the control (c) and ⊕ the target (t) bits.