PHYS-E0461 Introduction to Plasma Physics for Fusion and Space Applications
Second examination 12.12.2018

1. (a) Explain briefly how collisions differ in neutral gas, weakly ionized gas, and fully ionized plasma. (2p)
(b) Explain the major differences (advantages and disadvantages) between a tokamak and a stellarator as far as fusion energy production is concerned. (2p)
(c) Explain (you do NOT need to remember the names) the structure of Earth's atmosphere, i.e., on what basis the atmosphere is divided into different *spheres*. Also tell for which layers the name 'sphere' is not appropriate. (2p)

2. **Conductivities.** Using a simple steady-state fluid equation for the electrons, $0 = -e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nu_c m_e \mathbf{v}_e$, where ν_c is the collision frequency, and assuming stationary ions, show that the Ohm's law can be expressed as a matrix equation $\mathbf{J} = \sigma \cdot \mathbf{E}$, where

$$\sigma = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \quad (1)$$

when the z axis is taken along the constant magnetic field, $\mathbf{B} = B\hat{z}$. Here, $\sigma_P = \frac{\nu_c^2}{\nu_c^2+\Omega^2}\sigma_\parallel$ is the so-called *Pedersen conductivity*, $\sigma_H = \frac{\nu_c\Omega}{\nu_c^2+\Omega^2}\sigma_\parallel$ is the so-called *Hall conductivity*, $\sigma_\parallel = \frac{ne^2}{m_e\nu_c}$ is the parallel conductivity, and Ω the electron cyclotron frequency. What can you say about the plasma conductivity along the magnetic field compared to its value across the magnetic field? (6p)

3. **Ohmic heating.** Let's try to heat a tokamak plasma ohmically. Let the (constant) plasma current along B be $10^5 A/m^2$ and assume uniform plasma density at $n = 10^{19} m^{-3}$. The Joule heating, $dw/dt = \eta \cdot j^2$, goes predominantly to electrons. Here $w = n_e T_e$ is the electron energy density. Assuming the so-called *Spitzer* resistivity, $\eta = \frac{1}{32\pi\epsilon_0^2}\frac{e^2\sqrt{m_e}}{T_e^{3/2}}\ln\Lambda$, calculate the time dependence of electron temperature. If we start with T_e = 10 eV, how long does it take to reach 1 keV? How about the more fusion-relevant temperature of 10 keV? For the Coulomb logarithm, you can assume a constant value of $\ln\Lambda = 17$. (6p)

4. **Magnetosphere.** The solar wind exerts pressure on Earth's magnetosphere. For Earth to maintain its magnetosphere, its magnetic pressure *together with the plasma pressure* has to be able to stand up against solar wind's kinetic pressure. Using the pressure balance, derive a condition for the existence of a magnetosphere around Earth by assigning the magnetosphere an effective radius r_M. This has to be larger than the Earth radius, r_E for the magnetosphere to exist. (Hints: approximate Earth's magnetic field by its dipole field, $B \approx k_0/r^3$ with $k_0 \approx 10^{16}\text{Tm}^3$, and express the kinetic pressure of the solar wind as ρV^2, where ρ is the mass density of the solar wind and V its velocity. Also remember the parameter $\beta = p/(B^2/2\mu_0)$.) (4p)

5. **Inductive current drive.** In a tokamak the electric current is driven in the fully ionized plasma by an electric field applied along the magnetic field. This electric field is related to the *loop voltage*, $E = V_{loop}/2\pi R$, which is induced via transformer principle: Let the tokamak's central solenoid have n turns per unit length and have a current I_{coil} running in it. Then the (vertical) magnetic field inside the solenoid is $B = \mu_0 n I_{coil}$. For brevity, let's introduce a linear current density for the solenoid: $i = nI_{coil}$, $[i] = \text{A/m}$.
(a) Using Faraday's law (neglect the time derivative of electric field), show that the expression for the loop voltage is $V_{loop}(R) = -\pi\mu_0 R^2 \frac{di}{dt}$. (2p)
(b) At what rate should the current density in the central solenoid be ramped up to provide a tokamak like ASDEX Upgrade (minor radius a = 60 cm) a plasma current of 1 MA, if the temperature is T_e = 500 eV and the resistivity is given by Spitzer? For the major radius R you can use constant R = 1.75 m ('large' aspect ratio assumption) and take the plasma cross section as circular. Use a constant value of $\ln\Lambda = 17$ for the Coulomb logarithm. (6p)

1 Helpful physics formulas

Debye length	$\lambda_D^2 = \frac{\epsilon_0 T_e}{e^2 n_0}$	(2)
Plasma parameter	$\Lambda = \frac{4}{3} n_0 \pi \lambda_D^3$	(3)
Plasma frequency	$\omega_p^2 = \frac{e^2 n_0}{m_e \epsilon_0}$	(4)
Larmor frequency	$\Omega = \frac{qB}{m}$	(5)
Larmor radius	$r_L = \frac{m v_\perp}{qB}$	(6)
Magnetic moment	$\mu = \frac{m v_\perp^2}{2B}$	(7)
$\boldsymbol{E} \times \boldsymbol{B}$ drift	$\boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$	(8)
Gradient drift	$\boldsymbol{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\boldsymbol{B} \times \nabla B}{B^2}$	(9)
Diamagnetic drift	$\boldsymbol{v}_D = -\frac{\nabla p \times \boldsymbol{B}}{q n_0 B^2}$	(10)
Maxwell-Boltzmann	$f(\boldsymbol{v}) = \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right)$	(11)
	$h(E) = \frac{2}{T^{3/2}} \sqrt{\frac{E}{\pi}} e^{-E/T}$	(12)
Convective derivative	$\frac{\mathrm{d}n(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{\partial n(\boldsymbol{r},t)}{\partial t} + \boldsymbol{v} \cdot \nabla n(\boldsymbol{r},t)$	(13)
Collision frequency	$\nu = \frac{e^4 \ln \Lambda}{4\pi \epsilon_0^2 \sqrt{m}} \frac{n}{T^{3/2}}$	(14)
Gauss's law	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$	(15)
Gauss's law for magnetism	$\nabla \cdot \boldsymbol{B} = 0$	(16)
Maxwell-Faraday equation	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	(17)
Ampère's circuital law	$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$	(18)
Sound speed in ideal gases	$v_s = \sqrt{\frac{E_{\text{therm}}}{m}}$	(19)
Sound speed in plasma	$v_s = \sqrt{\frac{E_{\text{therm, e}}}{M} + \frac{\gamma_i E_{\text{therm, i}}}{M}}$	(20)

2 Constants

- vacuum permittivity $\epsilon_0 = 8.9 \cdot 10^{-12}$ C/Vm
- magnetic permeability $\mu_0 = 4\pi \times 10^{-7}$ Tm/A
- elementary charge $e = 1.6 \cdot 10^{-19}$ C
- electron mass $e = 9.1 \cdot 10^{-31}$ kg

3 Vector Identities

3.1 Identities Involving Only Vectors

(a) $\boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{B} \cdot \boldsymbol{C} \times \boldsymbol{A}$

(b) $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A} = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$

(c) $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$

(d) $(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = \boldsymbol{C}(\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{D}) - \boldsymbol{D}(\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C})$

3.2 Identities Involving ∇

(a) $\nabla \cdot (\psi \boldsymbol{A}) = \psi(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla \psi)$

(b) $\nabla \times (\psi \boldsymbol{A}) = \psi(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla \psi)$

(c) $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$

(d) $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A})$

(e) $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$

(f) $\nabla \cdot (\boldsymbol{AB}) = (\nabla \cdot \boldsymbol{A})\boldsymbol{B} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$

(g) $\nabla \cdot (\psi \overleftrightarrow{\boldsymbol{T}}) = \nabla \psi \cdot \overleftrightarrow{\boldsymbol{T}} + \psi \nabla \cdot \overleftrightarrow{\boldsymbol{T}}$

(h) $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

(i) $\nabla^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times \nabla \times \boldsymbol{A}$

(j) $\nabla(\psi\xi) = \nabla(\phi\xi) = \psi\nabla\xi + \xi\nabla\psi$

(k) $\nabla \cdot (\nabla\psi \times \nabla\xi) = 0$

(l) $\nabla \cdot \nabla\psi = \nabla^2\psi$

(m) $\nabla \times \nabla\psi = 0$

3.3 Identities Involving $\int$

(a) $\int_V \nabla\psi \, d\tau = \int_S \psi \, d\boldsymbol{\sigma}$

(b) $\int_V \nabla \times \boldsymbol{A} \, d\tau = \oint_S d\boldsymbol{\sigma} \times \boldsymbol{A}$

(c) $\int_S d\boldsymbol{S} \cdot \nabla \times \boldsymbol{A} = \oint_C d\boldsymbol{l} \cdot \boldsymbol{A}$

(d) $\oint_C d\boldsymbol{l} \times \boldsymbol{A} = \int_S (d\boldsymbol{S} \times \nabla) \times \boldsymbol{A}$

4 Curvilinear Coordinate Systems

4.1 Cylindrical Coordinates (r, θ, z)

(a) $\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$

(b) $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$

(c) $\nabla \times \boldsymbol{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial\psi}{\partial z}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial\psi}{\partial z}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{z}}$

(d) $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$

4.2 Spherical Coordinates (r, θ, ϕ)

(a) $\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$

(b) $\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

(c) $\nabla \times \boldsymbol{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$

(d) $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial^2\psi}{\partial d\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

5 Gaussian Integrals

Definite integral relations of Gaussian integrals

(a) $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2}$

(b) $\int_{-\infty}^\infty e^{-ax^2}dx = \left(\frac{\pi}{a}\right)^{1/2}$

(c) $\int_{-\infty}^\infty e^{-ax^2}e^{-2bx}dx = \left(\frac{\pi}{a}\right)^{1/2}e^{\frac{b^2}{a}}$ for $a > 0$

(d) $\int_{-\infty}^\infty xe^{-a(x-b)x^2}dx = b\left(\frac{\pi}{a}\right)^{1/2}$

(e) $\int_{-\infty}^\infty x^2e^{-ax^2}dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$

(f) $$\int_0^\infty x^n e^{-ax^2}dx = \begin{cases} \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)/a^{(n+1)/2} & a > 0 \\ \frac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\frac{\pi}{a}} & n = 2k, a > 0 \\ \frac{k!}{2a^{k+1}} & n = 2k+1, a > 0 \end{cases}$$