MS-E1462 Banach spaces (Aalto University)
Turunen / Vuojamo

Examination (Wednesday 19.12.2018, 13:00–16:00)

Points also for good effort! **Calculators and literature forbidden.**

Remark. In the following problems, X is a complex Banach space.
If using some results from the course, please state them.

1. Prove that in a normed space the closure and the interior of a convex set are convex.

2. Let $A : V \to X$ be a bounded linear mapping, where V is a dense vector subspace of a Banach space X. Show that there is a unique bounded linear mapping $B : X \to X$ such that $Au = Bu$ for all $u \in V$. Moreover, show that $\|B\| = \|A\|$.

   (Hint: Suppose $V \ni u_k \to u \in X$ and $V \ni v_k \to u \in X$. Show that $(Au_k)_{k=1}^{\infty}, (Av_k)_{k=1}^{\infty}$ are Cauchy sequences converging to the same limit $Bu \in X$. Check that this defines $B \in \mathscr{B}(X)$, and...)

3. Formulate (but do not prove) the following theorems:

   a) *Closed Graph Theorem.*

   b) *Hahn–Banach Theorem* about extending linear functionals.

   c) *Riesz Compactness Theorem* about closed unit balls in normed spaces.

4. Let X be a Banach space (over the field $\mathbb{C}$), $A \in \mathscr{B}(X)$, and $\lambda \in \mathbb{C}$. Define $B_n \in \mathscr{B}(X)$ by

$$B_n := (\lambda I - A) \sum_{k=0}^{n} (A/\lambda)^k.$$

   Simplify the formula of B_n. Assuming $|\lambda| > \|A\|$, find the limit $\lim_{n\to\infty} B_n$. What does this limit tell about the spectrum of A?