Mathematics
Aalto University
Hannukainen / Perkkiö

MS-E1651 - Numerical Matrix Computations

Exam 10.12.2018

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

Solve all problems. You have two options:

- Grade is based only on the exam.
- Grade is based on exercise points and exam points (40/60).

Indicate the preferred option clearly in the exam sheet.

1. (a) Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4+\frac{1}{4} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 5+\frac{1}{4} \end{bmatrix}.$$

Compute the Cholesky decomposition of A and use it to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$.

(b) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix such that

$$A = \begin{bmatrix} 1 & \boldsymbol{a}_{21}^T \\ \boldsymbol{a}_{21} & A_{22} \end{bmatrix} \quad \text{where } \boldsymbol{a}_{21} \in \mathbb{R}^{n-1} \text{ and } A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Find the lower triangular matrix $L \in \mathbb{R}^{n\times n}$ such that

$$A = L \begin{bmatrix} 1 & 0 \\ 0 & A_{22} - \boldsymbol{a}_{21}\boldsymbol{a}_{21}^T \end{bmatrix} L^T$$

Explain how this equation can be used to compute the Cholesky Decomposition of A.

2. (a) Let $a, b \in \mathbb{R}$ and consider the equation: Find $t \in \mathbb{R}$ such that

$$at = b$$

Assume, that t is solved in floating point arithmetics. Find $\Delta a \in \mathbb{R}$ such that

$$(a + \Delta a) fl(t) = b.$$

Explain, how the error $|t - fl(t)||t|^{-1}$ can be estimated.

(b) Let $a, b, c \in \mathbb{R}$. Show that

$$-\left|\frac{ab}{c}\right|(2u+u^2) \le fl\left(\frac{ab}{c}\right) - \frac{ab}{c} \le \left|\frac{ab}{c}\right|(2u+u^2).$$

where u is the machine epsilon.

3. (a) Let $\boldsymbol{x} \in \mathbb{R}^2$. Find the rotation matrix $G \in \mathbb{R}^{2\times 2}$ such that

$$G\boldsymbol{x} = \begin{bmatrix} \|\boldsymbol{x}\|_2 \\ 0 \end{bmatrix}$$

(b) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \pi \\ 1 & 0 \end{bmatrix}$$

Compute the QR-decomposition of A using Givens rotation matrices.

4. Let $\boldsymbol{b} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. In addition, let $J : \mathbb{R}^n \to \mathbb{R}$ be such that

$$J(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}.$$

(a) Let $\boldsymbol{x} \in \mathbb{R}^n$ be such that $A\boldsymbol{x} = \boldsymbol{b}$. Show that $J(\boldsymbol{x}) < J(\boldsymbol{x}+\boldsymbol{v})$ for every $\boldsymbol{v} \in \mathbb{R}^n, \boldsymbol{v} \neq 0$.

(b) Consider minimization of J using the line search method starting from initial guess $\boldsymbol{x}_0$. Denote iterates as $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ and the search direction on step n as $\boldsymbol{p}_n$.
Derive a formula for computing $\boldsymbol{x}_{n+1}$ from $\boldsymbol{x}_n$ and explain how conjugate gradient method is related to a line search method