

You are allowed to bring to the exam a handwritten **memory aid sheet**. The memory aid sheet must be of size A4 with text only on one side, and it must contain your name and student number in the upper right corner. You don't need to return your memory aid sheet. The exam consists of 4 problems, each worth 6 points in total.

The grading of the course is based on either 100% exam score or 50% exam score + 40% homeworks score + 10% quizzes score, whichever is higher.

1. Recall that a real valued random variable X is said to have probability density $f_X: \mathbb{R} \rightarrow [0, +\infty]$, if for all Borel sets $B \subset \mathbb{R}$ we have $P[X \in B] = \int_{\mathbb{R}} \mathbb{I}_B f_X d\Lambda$, and that an \mathbb{R}^2 -valued random vector Z is said to have probability density $f_Z: \mathbb{R}^2 \rightarrow [0, +\infty]$, if for all Borel sets $A \subset \mathbb{R}^2$ we have $P[Z \in A] = \int_{\mathbb{R}^2} \mathbb{I}_A f_Z d\Lambda^2$, where Λ is the Lebesgue measure on \mathbb{R} and $\Lambda^2 = \Lambda \otimes \Lambda$ is the 2-dimensional Lebesgue measure on \mathbb{R}^2 , and \mathbb{I}_S denotes the indicator function of a set S .

- (a) Give an example of a real valued random variable which has a probability density, and an example which does not have a probability density. **(1 p)**
- (b) Suppose that $Z = (X, Y)$ is an \mathbb{R}^2 -valued random vector which has probability density f_Z . Show that the components X and Y have probability densities f_X and f_Y given, respectively, by **(3 p)**

$$f_X(x) = \int_{\mathbb{R}} f_Z(x, y) d\Lambda(y) \quad \text{and} \quad f_Y(y) = \int_{\mathbb{R}} f_Z(x, y) d\Lambda(x).$$

- (c) Give an example of an \mathbb{R}^2 -valued random vector $Z = (X, Y)$ such that Z does *not* have a probability density, but its components X and Y both have probability densities. **(2 p)**

2. Let X and Y , and X_1, X_2, X_3, \dots , and Y_1, Y_2, Y_3, \dots be real valued random variables defined on a common probability space (Ω, \mathcal{F}, P) . Prove the following statements:

- (a) If we have $X_n \xrightarrow{\text{a.s.}} X$ as $n \rightarrow \infty$, then also $X_n \xrightarrow{P} X$ as $n \rightarrow \infty$. **(2 p)**
- (b) If we have $X_n \xrightarrow{\text{a.s.}} X$ and $Y_n \xrightarrow{\text{a.s.}} Y$ as $n \rightarrow \infty$, then also $X_n + Y_n \xrightarrow{\text{a.s.}} X + Y$ as $n \rightarrow \infty$. **(2 p)**
- (c) If we have $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ as $n \rightarrow \infty$, then also $X_n + Y_n \xrightarrow{P} X + Y$ as $n \rightarrow \infty$. **(2 p)**

3. Suppose that P_1 and P_2 are two probability measures on the same measurable space (Ω, \mathcal{F}) . Define the collection

$$\mathcal{C} = \{A \in \mathcal{F} \mid P_1[A] = P_2[A]\}.$$

- (a) Prove that the collection satisfies the following properties: (4 p)

- (i): $\Omega \in \mathcal{C}$
- (ii): if $A, B \in \mathcal{C}$ and $A \subset B$, then $B \setminus A \in \mathcal{C}$
- (iii): if $A_1, A_2, \dots \in \mathcal{C}$ and $A_n \uparrow A$, then $A \in \mathcal{C}$.

- (b) State the Dynkin's identification theorem. Provide also all the necessary additional definitions needed in the statement, which go beyond the notions appearing in part (a) (i.e., beyond those that concern $\Omega, \mathcal{F}, P_1, P_2, \mathcal{C}$). (2 p)

4. Suppose that X is a real valued random variable with standard normal distribution, i.e., a continuous distribution with density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{for } x \in \mathbb{R}.$$

Let $m \in \mathbb{R}$ and $s > 0$, and let $Y = m + sX$. Let also

$$\varphi_X(\theta) = E[e^{i\theta X}] \quad \text{and} \quad \varphi_Y(\theta) = E[e^{i\theta Y}].$$

Hint: In this problem, you can consider it known that $\int_{-\infty}^{+\infty} f_X(x) dx = 1$, and that the exponential of any complex number $z \in \mathbb{C}$ is given by the convergent series $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$.

- (a) Show that $\varphi_Y(\theta) = e^{im\theta} \varphi_X(s\theta)$ for all $\theta \in \mathbb{R}$. (1 p)

- (b) Let $t \in \mathbb{R}$. Show that $E[e^{tX}] = e^{t^2/2}$. (1 p)

Hint: Express the expected value in terms of the density, and perform a suitable change of variables $x' = x + c$.

- (c) For $x, t \in \mathbb{R}$, show that $e^{|tx|} \leq e^{tx} + e^{-tx}$. Use this and earlier results to prove that for any $t \in \mathbb{R}$ we have (1 p)

$$E\left[\sum_{n=0}^{\infty} \frac{1}{n!} |tX|^n\right] < +\infty.$$

- (d) Prove that for any $t \in \mathbb{R}$ we have (1 p)

$$E[e^{tX}] = \sum_{n=0}^{\infty} \frac{1}{n!} t^n E[X^n].$$

- (e) By comparing (b) with (d), deduce that for $n \in \mathbb{N}$ we have (1 p)

$$E[X^n] = \begin{cases} \prod_{j=1}^{n/2} (2j-1) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

- (f) Prove that $\varphi_X(\theta) = e^{-\theta^2/2}$ for all $\theta \in \mathbb{R}$. (1 p)