Question 1:

(a) Briefly explain what a group is. (3)

(b) For an element x of a group G, explain what is meant by the order of x. (3)

(c) Write down the definition of a subgroup H of a group G, and describe the shortest criterion to be checked in order to verify the subgroup property. (4)

Question 2: Write down a Cayley table of the group $\mathbb{Z}_8^\times$, and determine, whether or not this group is a cyclic group. (10)

Question 3: Let G be a group, and let S be a set of generators of G. Assume that $xy = yx$ for all $x, y \in S$. Prove that G must be abelian. This means, to test whether or not a group is abelian, it suffices to verify the commutative rule on a set of generators. (10)

Question 4: Let G and H be cyclic groups such that G has k elements, and H has n elements. Show that if $\gcd(k, n) = 1$, then $G \times H$ is again cyclic. (10)
Hint: Find an element in $G \times H$ that is of order kn.

Question 5: Let G be a group. Prove that the mapping $x \mapsto x^{-1}$ is an automorphism of G if and only if G is abelian. (10)

Question 6: Let $\mathbb{C}^\times$ denote the multiplicative group of non-zero complex numbers, and let $\mathbb{R}^\times$ denote the same for real numbers. Consider the absolute value function $\mathbb{C}^\times \longrightarrow \mathbb{R}^\times$, $z \mapsto |z|$.

(a) Briefly verify that this function is a homomorphism. (4)

(b) Determine the kernel of this homomorphism. (6)

Question 7: Let $\varphi : G \longrightarrow H$ be a group homomorphism. Show that if G is cyclic, then $\mathrm{im}(\varphi)$ is cyclic. (5)

Question 8: Let $\varphi : G \longrightarrow G'$ be a surjective group homomorphism, and let H be a normal subgroup of G. Show that $\varphi(H)$ is a normal subgroup of G'. (10)

Question 9: Let G be a group and let $x_1, \dots, x_r$ be a set of generators of G, which means $G = \langle x_1, \dots, x_r \rangle$. Finally, let H be a subgroup of G.

(a) Assuming $x_i^{-1} H x_i = H$ for all $i = 1, \dots, r$, show that H is normal in G. (6)

(b) Suppose G is finite, and assume $x_i^{-1} H x_i \subseteq H$ for all $i = 1, \dots, r$. Show that H is normal in G. (7)

(c) Again, assume that G is finite, and that H is generated by elements $y_1, \dots, y_m$. Assume that $x_i^{-1} y_j x_i \in H$ for all $i = 1, \dots, r$ and $j = 1, \dots, m$. Show that H is normal in G. (7)

Question 10: Let p be a prime number, and let R be the subset of all rational numbers $\frac{m}{n}$ such that $n \neq 0$ and n is not divisible by p. Show that R is a unital ring. (10)

Question 11: Let R be an integral domain, and let $0 \neq a \in R$.

(a) Show that the mapping $x \mapsto ax$ is an injective mapping of R into itself. (5)

(b) Use this in order to show, that if R is finite, then R is a field. (5)

Question 12: Let R be a commutative ring. Take it as proven, that the binomial theorem holds, which means that for arbitrary $a, b \in R$ and $n \in \mathbb{N}$ there holds

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

The expressions $\binom{n}{i}$ have to be read as $\binom{n}{i}$-fold sums of the identity 1, so that these are now elements of R. An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some positive integer n. Show the following:

(a) If $a \in R$ is nilpotent, then $1 + a$ and $1 - a$ are invertible elements. (6)
Hint: Recall a well-known expression for $\frac{1}{1-x}$ and adapt it to this situation.

(b) The set $N(R)$ of all nilpotent elements in R forms a (two-sided) ideal of R. (6)

(c) The only nilpotent element of $R\,/\,N(R)$ is its zero element $N(R)$. (8)