Aalto University
School of Science

PHYS-E0416 Quantum Physics
Midterm Exam 22.02.2019

1. Explain briefly the following concepts. (You may use equations if you find it convenient, but it is not necessary.)

   a) Dipole approximation

   b) What do we know about symmetry properties of quantum many-body states and to what kind of classification of particles this leads to?

   c) Spontaneous emission and its connection to quantizing the electromagnetic field

2. Derive the Fermi golden rule for the (absorption) transition rate from the initial state $|\psi(t=0)\rangle = |i\rangle$ to a continuous part of the spectrum.

   a) Start from the first-order time-dependent perturbation theory result

   $$\langle f|\psi(t)\rangle \approx \frac{1}{\hbar}\int_0^t dt' \, \langle f|H'(t')|i\rangle e^{i\omega_{fi}t'},$$

   where $|i\rangle$ and $|f\rangle$ are the initial and final states of the system, $\omega_{fi} = (E_f - E_i)/\hbar$, and $H'(t) = Ae^{-i\omega t} + A^\dagger e^{i\omega t}$ is a harmonic perturbation. Derive the result for resonant absorption probability

   $$P_{if}(t) = \frac{1}{\hbar^2}|\langle f|A|i\rangle|^2 \left[\frac{\sin\left[\frac{1}{2}(\omega-\omega_{fi})\,t\right]}{\frac{1}{2}(\omega-\omega_{fi})}\right]^2, \quad \text{for} \quad \omega \approx \omega_{fi},$$

   You can assume that the perturbation is long lasting, such that $|\omega t| \gg 1$.

   b) Using the above transition probability, write the total transition rate out from the initial state $|i\rangle$ when there are several possible final states $|f_n\rangle$ with energies $E_{f,n} = \hbar\omega_n$. You can assume that the coupling $\langle f|A|i\rangle$ is equal for all final states $|f\rangle = |f_n\rangle$. Now write the total transition rate when the final states form a continuum with constant density of states $g(E) = g$.

   c) Solve the total transition rate and derive the Fermi golden rule. Since off-resonant (non-energy-conserving) processes are strongly suppressed, you are able to do some approximations. The following relation may also prove useful

   $$\int_{-\infty}^{\infty}\left[\frac{\sin x}{x}\right]^2 dx = \pi.$$

3. The Hamiltonian in the field operator formulation is

$$\begin{aligned} H &= \int d^3x \left(\frac{\hbar^2}{2m}\nabla\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}) + U(\mathbf{x})\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\right) \\ &+ \frac{1}{2}\int d^3x \int d^3x' \, \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')V(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x}), \end{aligned}$$

where U is an external potential, m the mass of the particle, and V the interaction potential. Starting from the Heisenberg equation of motion

$$\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -[H,\psi(\mathbf{x},t)] = -e^{iHt/\hbar}[H,\psi(\mathbf{x},0)]e^{-iHt/\hbar}$$

derive the equation of motion for the field operator:

$$\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{x})\right)\psi(\mathbf{x},t) + \int d^3x'\psi^\dagger(\mathbf{x}',t)V(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}',t)\psi(\mathbf{x},t). \quad (1)$$

How does the form of this equation relate to the name "second quantization"?

4. Consider the coherent state for which $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\hat{a}$ is the annihilation operator for a photon. Calculate the fluctuations $\Delta A = \sqrt{\langle\alpha|\hat{A}^2|\alpha\rangle - \langle\alpha|\hat{A}|\alpha\rangle^2}$, when the operator $\hat{A}$ is

   a) Photon number $\hat{n}$
   b) The quadrature $\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger)$
   c) The quadrature $\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger)$

   By computing $\Delta q\Delta p$ show that coherent state is a minimum uncertainty state. In case of vacuum, $\alpha = 0$, how does your result for Δn reflect energy conservation?