Mathematics
Aalto University

Hannukainen/Kuutela

MS-E1653 Finite Element Method

Exam 12.4.2019

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark the course code, title and either the text final exam (option 1) or course exam (option 2).

You have two options:

1. Solve all four problems. The grade is based only on the exam.
2. Solve any three problems. The grade is based on exercise points + exam points. To choose this option, you must have participated to the course during spring 2019 and completed the final project.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Let Ω be a bounded domain and $f \in L^2(\Omega)$. Consider the strong problem: find u such that

$$\begin{cases} -\nabla \cdot (\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

(a) Derive the weak form of the strong problem (1) : find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V.$$

Specify a, L, the space V and it's norm $\|\cdot\|_V$.

(b) Show that there exists $C_1, C_2 > 0$ independent on u and v such that

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_V \quad \text{and} \quad |L(v)| \leq C_2 \|v\|_V \quad \forall u, v \in V.$$

(c) Formulate the Poincare - inequality and show that there exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in V.$$

2. Let $V = \text{span}\{\varphi_i\}_{i=1}^n$ for some basis $\{\varphi_i\}_{i=1}^n$, $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that L is bounded and a is continuous, symmetric and coercive.

(a) Show that the problem: find $u \in V$ such that

$$a(u,v) = L(v), \quad \forall v \in V,$$

is equivalent to: find $u \in V$ such that

$$a(u,\varphi_j) = L(\varphi_j), \quad j = 1,\ldots,n.$$

(b) Expand $u = \sum_{j=1}^n \beta_j \varphi_j$ and show that $\boldsymbol{\beta}$ is solution to: Find $\boldsymbol{\beta} \in \mathbb{R}^n$ such that

$$A\boldsymbol{\beta} = \boldsymbol{b},$$

where $A \in \mathbb{R}^{n \times n}, \boldsymbol{b} \in \mathbb{R}^n$ are such that $A_{ij} = a(\varphi_j, \varphi_i)$ and $\boldsymbol{b}_i = L(\varphi_i)$ for $i,j = 1,\ldots,n$.

(c) Show that $A = A^T$.

3. Let the finite element mesh $\mathcal{T}$ be such that

$$p = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 5 \\ 4 & 5 & 5 & 6 \end{bmatrix} \tag{2}$$

(a) Draw the mesh $\mathcal{T}$.

(b) Compute affine mapping from the reference element to elements 3 and 4.

(c) Consider the bilinear form $a(u, v) = (\nabla u, \nabla v)$ and assume that standard linear nodal basis functions are used. Compute the row 3 of the system matrix

4. Let $V \subset H^1(\Omega)$, $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that L is bounded and a is continuous, symmetric and coercive. Consider the problem : find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{3}$$

Let $V_h \subset V$. The finite element approximation to problem (2) is : find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

(a) Formulate and prove Galerkin orthogonality property.

(b) Formulate and prove Céa's Lemma.