MS-C2105 - Introduction to optimization **Fabricio Oliveira**
Final Exam **9.4.2019**

The exam consists of 4 problems and is three hours long. The exam is graded on a scale of 0-100 points. The points assigned to each question are indicated in parenthesis within the text. Read the questions carefully and make sure your answers are clear, readable, and well organised. Good luck!

Problem 1 (25pt)

Consider the following linear programming problem:

$$\begin{aligned} P_1 : \text{max. } z_1 = \ & 3x_1 + 2x_2 \\ \text{s.t.: } & 4x_1 + 2x_2 \leq 6 \\ & 2x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

a. (10pt): Solve this problem using the Simplex method. You are required to:

1. Provide the problem in the standard form and the initial basis.
2. Indicate the variables becoming basic/ nonbasic at each iteration and why.
3. Provide the system of equations OR table form for each basis considered in the progress of the algorithm.
4. Indicate the optimal solution (x_1^*, x_2^*) and optimal objective function value z_1^*.
5. Indicate which constraints are active and inactive at the optimum.

Hint: you only need 2 iterations. You can choose to use either the table representation OR the systems of equations.

b. (5pt): Provide a graphical representation of problem P_1 in **(a.)**. You are required to:

1. Indicate the feasible region.
2. Indicate the points (x_1, x_2) that the method visited at each iteration (including the starting point).

c. (10pt): Show how the information from the optimal basis obtained in **(a.)** can be used to solve P_2 by answering the following:

$$\begin{aligned} \text{min. } P_2 : z_2 = \ & 6y_1 + 4y_2 \\ \text{s.t.: } & 4y_1 + 2y_2 \geq 3 \\ & 2y_1 + 2y_2 \geq 2 \\ & y_1, y_2 \geq 0. \end{aligned}$$

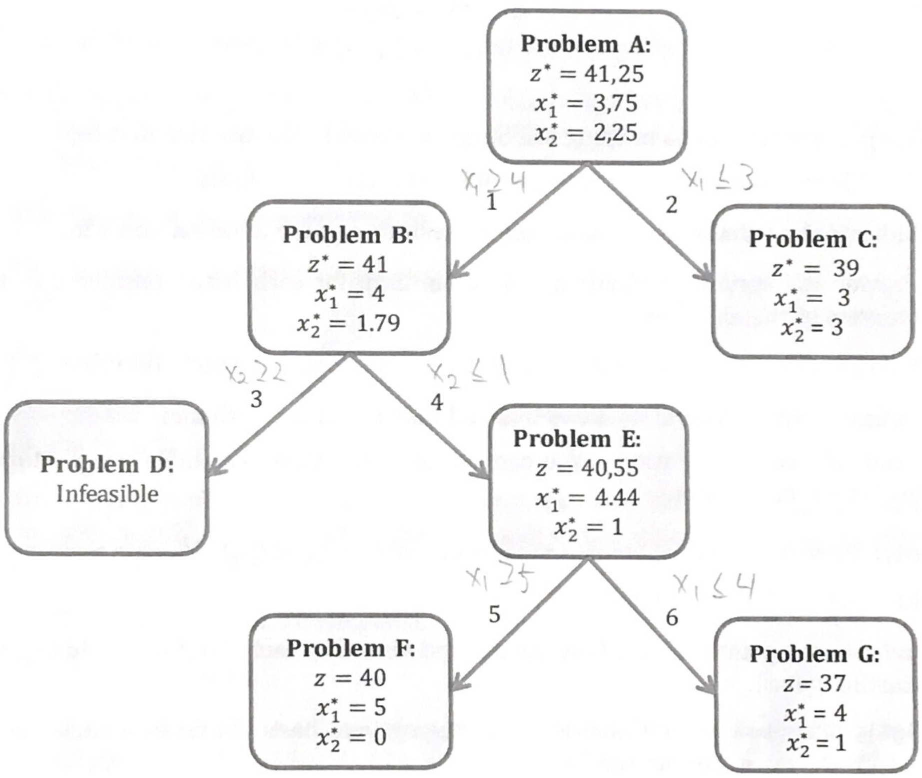
1. What is the relationship between P_1 and P_2? Justify your answer.
2. What result allows you to draw your conclusions? (no proofs required; it suffices to indicate which result and its consequences).
3. What is the optimal point (y_1, y_2) and optimal objective function value z_2^*, according to the information from the optimal basis obtained in **(a.)**.

Problem 2 (15pt)

Considering the following IP problem:

$$
\begin{aligned}
\text{max. } z = {} & 8x_1 + 5x_2 \\
\text{s.t.: } & x_1 + x_2 \leq 6 \\
& 9x_1 + 5x_2 \leq 45.
\end{aligned}
$$

After applying the branch-and-bound method, the following tree was obtained.



The arrows represent the derivation of each subproblem from its parent problem. The numbers above the arrows represent the constraints added to the parent problem to generate each subproblem. The variable selection was performed picking the variable with smallest index (e.g., in case x_1 and x_2 are fractional, x_1 is selected). Please answer the following:

a. (6pt): Identify what are the constraints represented by numbers 1, 2, 3, 4, 5, and 6.

b. (4pt): Which node represent the optimal solution for the IP problem? Please justify.

c. (5pt): Consider the possible sequences in which the subproblems B to G have been solved. Is there a sequence in which some of the subproblems do not need to be solved? Justify your answer.

Problem 3 (25pt)

A chemical plant can purchase up to 15 kg of a chemical for \$10/kg. At a cost of \$3/kg, the chemical can be processed into 1kg of product A; or at a cost of \$5/kg, the chemical can be processed into one kg of product B. If x_1 kg of product A are produced, it sells for a price of $\$(30 - x_1)$ per kilo. If x_2 kg of product B are produced, it sells for a price of $\$(50 - 2x_2)$ per kilo. Let y kg be the amount of the chemical purchased. The model that optimise the profit of the chemical plant is given by:

$$
\begin{aligned}
P : \max.\ z = {} & x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 - 10y \\
\text{s.t.: } & x_1 + x_2 \leq y \\
& y \leq 15 \\
& x_1, x_2, y \geq 0.
\end{aligned}
$$

a. (6pt): Formulate the Karush-Kuhn-Tucker (KKT) optimality conditions for the above problem P.

b. (4pt): Are the KKT conditions sufficient for optimality in this case? If so, would a point satisfying these conditions be a local or global optimum? Please justify your answers.

c. (5pt): Solve the conditions formulated in **(a.)** to obtain a point $x = (x_1, x_2, y)$ satisfying the KKT conditions. **Hint:** start with the cases in which x_1, x_2 and y are all greater than zero and thus $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

d. (5pt): Consider the following situation: a third product, product C, is incorporated as an alternative in the production process. At a cost of \$6/kg, the chemical can be processed into 2kg of product C. Reformulate the model P that optimise the profit of the chemical plant, incorporating the option of producing product C. You are required to identify any elements you may want to include or modify in problem P. C sells for \$ 10/kg

e. (5pt): Consider the following situation: suppose that, due to technical reasons, either product A or product B can be produced, but not both simultaneously. Reformulate the original model P (not the one you formulated in **(d.)**) to include this situation. You are required to identify any elements you may want to include or modify in problem P.

linear constraints

Problem 4 (35pt)

Consider the following function:

$$f(x_1, x_2) = (2 - x_1)^2 + (2 - 2x_1 - x_2)^2$$

a. (5pt): Obtain the optimum/ optima for this function analytically. Please provide:

1. The optimality conditions used to find the candidate point(s).
2. The candidate point(s) obtained.
3. Arguments supporting whether the point(s) is(are) locally or globally optimal.

b. (15pt): Apply a single iteration of the gradient method to find an optimum for this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an optimal step size λ. You are requested to provide:

1. The expression for the gradient step.
2. The calculations for the optimal step size (analytically). **Hint:** should be a value close to 0.1).
3. The new point found.
4. Answer the following: is this point optimal? Please justify without relying on the results from **(a.)**.

c. (15pt): Apply a single iteration of the Newton's method to find the optimal of this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an a step size of $\lambda = 1$. You are requested to provide:

1. The expression for the Newton step.
2. The new point found.
3. Answer the following: is this point optimal? If so, why did the method only took a single iteration? Please justify without relying on the results from **(a.)**.

Hint: Remind that if $g(x) = (f(x))^n$ then the derivative is $g'(x) = nf(x)^{n-1}f'(x)$. Also, you need this result for **(c.)**: $\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix}$.