

PHYS-E0416 Quantum Physics
2nd exam 10.4.2019

1. Write a mini-essay (0.5-1 page each) about the following two topics:
 - a) Essentials of Fermi liquid theory
 - b) Berry curvature and quantum metric
2. Consider non-interacting bosons in a cube of volume $V = L^3$, in which L is the length of the cube. Show that there is Bose-Einstein condensation for certain particle numbers and temperatures.

Guidelines: The bosons are described by the Hamiltonian

$$H = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \psi(\mathbf{x}). \quad (1)$$

Assuming a periodic boundary condition, the single particle eigenstates are

$$\varphi_{\mathbf{n}}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_{\mathbf{n}} \cdot \mathbf{x}} \quad (2)$$

with the wave-vector $\mathbf{k}_{\mathbf{n}} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_i = 0, \pm 1, \pm 2, \dots$. At an inverse temperature $\beta = \frac{1}{k_B T}$ and chemical potential μ the expected number of particles in state $\varphi_{\mathbf{n}}$ is

$$n_{\mathbf{n}} = \langle \hat{a}_{\mathbf{n}}^\dagger \hat{a}_{\mathbf{n}} \rangle = \text{Tr} \{ \hat{\rho} \hat{a}_{\mathbf{n}}^\dagger \hat{a}_{\mathbf{n}} \} = \frac{z e^{-\beta E_{\mathbf{n}}}}{1 - z e^{-\beta E_{\mathbf{n}}}}, \quad (3)$$

with the notation $z = e^{\beta\mu}$. The total density of particles in the system is

$$\frac{N}{V} = \frac{1}{V} \sum_{\mathbf{n}} n_{\mathbf{n}} = \frac{1}{V} \sum_{\mathbf{n}} \frac{z e^{-\beta E_{\mathbf{n}}}}{1 - z e^{-\beta E_{\mathbf{n}}}}. \quad (4)$$

You may find it useful to estimate a discrete sum with an integral, to employ spherical coordinates and to use the following function

$$g_{\frac{3}{2}}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \frac{z e^{-x^2}}{1 - z e^{-x^2}}, \quad g_{\frac{3}{2}}(1) = 2.612 \dots \quad (5)$$

3. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i$$

together with the Gutzwiller mean-field ansatz

$$|\Psi_{MF}\rangle = \prod_{i=1}^M \left[\sum_{n_i=0}^{\infty} f_{n_i}^{(i)} |n_i\rangle \right].$$

a) Calculate the following quantities:

$$\begin{aligned} & \langle \Psi_{MF} | H | \Psi_{MF} \rangle, \\ & \langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle, \\ & \langle \Psi_{MF} | a_i | \Psi_{MF} \rangle, \\ & \sigma_i^2 = \frac{\langle \Psi_{MF} | \hat{n}_i^2 | \Psi_{MF} \rangle - \langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle^2}{\langle \Psi_{MF} | \hat{n}_i | \Psi_{MF} \rangle}. \end{aligned}$$

(We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

b) Explain briefly how you can distinguish between a Mott insulator and a superfluid phase based on the quantities above.

4. Consider the mean-field BCS Hamiltonian

$$H = \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}. \quad (6)$$

Diagonalize this by using the Bogoliubov transformation. Calculate the eigenenergies. You do not need to calculate the eigenvector amplitudes, we give them here:

$$u_{\mathbf{k}} = u_{\mathbf{k}\uparrow} = v_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2} \left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} \right)}, \quad (7)$$

$$v_{\mathbf{k}} = v_{\mathbf{k}\uparrow} = -u_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2} \left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} \right)}, \quad (8)$$

while the matrix for eigenvectors is

$$U = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix}. \quad (9)$$

How are the new quasiparticle operators $\gamma_{\mathbf{k}\uparrow}$, $\gamma_{\mathbf{k}\downarrow}$ related to the original operators $\alpha_{\mathbf{k}\uparrow}$, $\alpha_{\mathbf{k}\downarrow}$? Show that the quasiparticle operators obey the same (anti)-commutation relations as the original operators.