MS-E1993 Time-Frequency Analysis (Aalto University)
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Examination (Friday 7.7.2017, 10:15-13:15)

Points also for good effort! **Calculators and literature forbidden.**
Remark: The *ambiguity function* $\chi[u] = \chi(u,u)$ is a special case of the *ambiguity transform* $\chi(u,v)$ of nice-enough signals $u, v : \mathbb{R} \to \mathbb{C}$, defined by

$$\chi(u,v)(\xi, y) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi x \cdot \xi}\, u(x+y/2)\, v(x-y/2)^*\, \mathrm{d}x. \qquad (1)$$

In your solutions, feel free to apply changes of variables, like

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(a,b)\,\mathrm{d}a\,\mathrm{d}b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(x+y/2, x-y/2)\,\mathrm{d}x\,\mathrm{d}y.$$

1. How do we define the Wigner transform? Show that the ambiguity transform is the symplectic Fourier transform of the Wigner transform.

2. For $v \in H = L^2(\mathbb{R})$, let $Pv := \langle v, w\rangle w$, where $w \in H$, $\|w\| = 1$. Define the symmetric time-frequency transform C at $(0,0) \in \mathbb{R} \times \widehat{\mathbb{R}}$ by

$$C(u,v)(0,0) := \langle u, Pv\rangle.$$

   Show that $C[u] = C(u,u)$ is the spectrogram corresponding to the w-windowed Short-Time Fourier Transform.

3. The Born–Jordan transform $Q(u,v)$ is defined by

$$Q(u,v)(x,\eta) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \frac{1}{y} \int_{x-y/2}^{x+y/2} u(t+y/2)\, v(t-y/2)^*\, \mathrm{d}t\, \mathrm{d}y.$$

   a) Find $Q[u] = Q(u,u)$ for $u(t) = \mathrm{e}^{\mathrm{i}2\pi t \cdot \alpha}$, where $\alpha \in \mathbb{R}$.

   b) Find $Q[v] = Q(v,v)$ for $v(t) = 2\cos(2\pi t) = \mathrm{e}^{\mathrm{i}2\pi t} + \mathrm{e}^{-\mathrm{i}2\pi t}$ (write your solution clearly real-valued).

4. Time-frequency transform C is *positive* if $C[u] \geq 0$ for test signals u. Quantization $a \mapsto a_C$ is called *positive* if $\langle u, a_C u\rangle \geq 0$ for test signals u and positive symbols $a \geq 0$. Show that time-frequency transform C is positive if and only if the quantization $a \mapsto a_C$ is positive.
   (Remark: $a \geq 0$ accurately means "$a(x,\eta) \geq 0$ for almost all (x,η)".)