FINAL EXAM, DIFFERENTIAL AND INTEGRAL CALCULUS I, MS-A0111

- Time: 25.10.2018, 9:00-12:00
- Equipment: None
- Answer each problem on a separate sheet. Each problem is worth 4 points.
- Motivate all solutions carefully. Answers without motivation give no points.
- Good luck, have fun! /Ragnar

Problem 1

Compute the limit

$$\lim_{n\to\infty} n\sqrt{1+\frac{1}{n}} - \sqrt{n^2-1}.$$

Problem 2

Let the number sequence (a_n) be recursively defined by $a_0 = 1$ and

$$a_n = a_{n-1}\left(\frac{n}{n+1}\right)^n \quad \text{if } n \geq 1.$$

a) Compute the terms a_1, a_2, a_3.

b) Does the series

$$\sum_{n=0}^{\infty} a_n$$

diverge or converge?

Problem 3

Sketch the graph of

$$f(x) = \sin x(1+\sin x) \text{ on } [0, 2\pi].$$

Problem 4

Compute

$$\int_0^{\pi} \cos x \cdot \sin^2 x \; dx.$$

Problem 5

Compute

$$\int_0^1 \frac{x^4}{x^3-1} \; dx.$$

Problem 6

Solve the initial value problem

$$\begin{cases} y' &= y^2+1 \\ y(0) &= 0. \end{cases}$$

Standard limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0}\frac{\ln(x+1)}{x} = 1$$
$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$
$$\lim_{n\to\infty}\frac{\ln n}{n^\alpha} = 0 \text{ if } \alpha > 0$$
$$\lim_{n\to\infty}\frac{n^\alpha}{\beta^n} = 0 \text{ if } \beta > 1$$
$$\lim_{n\to\infty}\frac{\beta^n}{n!} = 0$$
$$\lim_{n\to\infty}\frac{n!}{n^n} = 0$$

Standard derivatives

$f(x)$	$f'(x)$
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos(x)^2}$
e^x	e^x
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{1}{-\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$g(h(x))$	$g'(h(x))h'(x)$
$g(x)h(x)$	$g(x)h'(x)+g'(x)h(x)$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$

Inverse functions

$$\arcsin : [-1,1] \to [-\tfrac{\pi}{2},\tfrac{\pi}{2}]$$
$$\arccos : [-1,1] \to [0,\pi]$$
$$\arctan : \mathbb{R} \to [-\tfrac{\pi}{2},\tfrac{\pi}{2}]$$
$$\ln : (0,\infty) \to \mathbb{R}$$
$$\sqrt{\cdot} : [0,\infty) \to [0,\infty)$$

Trigonometric identities

$$\tan(x) = \frac{\sin x}{\cos x}$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\sin(x+y) = \sin x\cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\sin(x)^2+\cos(x)^2 = 1$$

θ	radians	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	$-$

Algebraic identities

$$a^{x+y} = a^x a^y$$
$$\ln(xy) = \ln x + \ln y$$
$$x^n - 1 = (x-1)(1+x+\cdots+x^{n-1})$$
$$(x+y)^2 = x^2+2xy+y^2$$

Newton Raphson update rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Taylor series

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad \text{if } -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x$$