

**FINAL EXAM, DIFFERENTIAL AND INTEGRAL CALCULUS I,
MS-A0111**

- Time: 13.12.2018, 9:00-12:00
- Equipment: None
- Answer each problem on a separate sheet. Each problem is worth 4 points.
- Motivate all solutions carefully. Answers without motivation give no points.
- Good luck, have fun! /Ragnar

PROBLEM 1

Compute the limit

$$\lim_{x \rightarrow 0} \sin(x) \sin\left(\frac{1}{x}\right),$$

or show that the limit does not exist.

PROBLEM 2

Find the maximum and minimum value on $[-\sqrt{2}, \sqrt{2}]$ of

$$f(x) = \sqrt{e^{x^3-x}}.$$

PROBLEM 3

Find a rational number $\alpha \approx \sin(1)$ such that $|\sin(1) - \alpha| < \frac{1}{1000}$.

PROBLEM 4

Compute

$$\int_0^1 \frac{x + \arctan x}{1 + x^2} dx.$$

PROBLEM 5

Compute

$$\int_0^1 \frac{x}{x^3 + x^2 + x + 1} dx.$$

PROBLEM 6

Solve the initial value problem

$$\begin{cases} y'' - y = \sin x \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

STANDARD LIMITS

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= 1 \\ \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} &= 1 \\ \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= 1 \\ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n &= e \\ \lim_{n \rightarrow \infty} \frac{\ln n}{n^\alpha} &= 0 \text{ if } \alpha > 0 \\ \lim_{n \rightarrow \infty} \frac{n^\alpha}{\beta^n} &= 0 \text{ if } \beta > 1 \\ \lim_{n \rightarrow \infty} \frac{\beta^n}{n!} &= 0 \\ \lim_{n \rightarrow \infty} \frac{n!}{n^n} &= 0 \end{aligned}$$

STANDARD DERIVATIVES

$f(x)$	$f'(x)$
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos(x)^2}$
e^x	e^x
$\ln x $	$\frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$g(h(x))$	$g'(h(x))h'(x)$
$g(x)h(x)$	$g(x)h'(x) + g'(x)h(x)$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$

INVERSE FUNCTIONS

$$\begin{aligned} \arcsin : [-1, 1] &\rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \arccos : [-1, 1] &\rightarrow [0, \pi] \\ \arctan : \mathbb{R} &\rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \ln : (0, \infty) &\rightarrow \mathbb{R} \\ \sqrt{\cdot} : [0, \infty) &\rightarrow [0, \infty) \end{aligned}$$

TRIGONOMETRIC IDENTITIES

$$\begin{aligned} \tan(x) &= \frac{\sin x}{\cos x} \\ \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \tan(-x) &= -\tan x \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x)^2 + \cos(x)^2 &= 1 \end{aligned}$$

θ	radians	sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	-

ALGEBRAIC IDENTITIES

$$\begin{aligned} a^{x+y} &= a^x a^y \\ \ln(xy) &= \ln x + \ln y \\ x^n - 1 &= (x-1)(1+x+\dots+x^{n-1}) \\ (x+y)^2 &= x^2 + 2xy + y^2 \end{aligned}$$

NEWTON RAPHSON UPDATE RULE

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

TAYLOR SERIES

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{if } -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \text{for all } x$$