MS-C2111 Stochastic processes
Department of Mathematics and Systems Analysis
Aalto University

Exam
29.5.2019, 16:30–19:30
L Leskelä & J Karjalainen

You are allowed to use a **calculator** approved by the Finnish Matriculation Examination Board and a handwritten **cheat sheet**. The cheat sheet must be of size a4 with text only on one side, and it must contain your name and student number in the upper right corner. You don't need to return the cheat sheet. The exam contains 4 problems, each worth 0–6 points.

1. A discrete-time Markov chain in state space $\{1,2,3,4,5\}$ has a transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/10 & 0 & 0 & 0 & 9/10 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Draw a transition diagram for the chain. (1 p)

(b) If the chain is now in state 2, then what is the probability that the chain is in state 2 also after three time steps? (1 p)

(c) Does the chain have (one or more) equilibrium distributions? If yes, find out all equilibrium distributions. (2 p)

(d) What is the probability that the chain started in state 2 reaches state 4 at some time instant? (2 p)

2. Let $(X_0, X_1, X_2, \ldots)$ be a discrete-time and time-homogeneous Markov chain with state space $\{1,2,3\}$ and transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Assume that the chain starts in state 1. Let $\tau = \min\{t \geq 0 : X_t = 3\}$ be the time that it takes for the chain to reach state 3. Determine:

(a) $\mathbb{P}(\tau = 1)$, (1 p)

(b) $\mathbb{P}(\tau = 2)$, (1 p)

(c) $\mathbb{P}(\tau = \infty)$, (2 p)

(d) $\mathbb{E}(\tau)$. (2 p)

3. Let $(M_0, M_1, M_2, \dots)$ be a time-homogeneous discrete-time Markov chain on state space $S = \{-1, 0, +1\}$ with transition matrix P and a random initial state M_0 which is uniformly distributed in S. A stranger on a street claims that (M_t) is a martingale with respect to its own information.

(a) Give an example of P for which the claim is true, and explain why. (2 p)

(b) Give an example of P for which the claim is not true, and explain why. (2 p)

(c) Give an example of P for which (M_t) is a submartingale but not a martingale. (2 p)

4. Ships arrive at a harbor at random time instants with independent exponentially distributed interarrival times at an average rate of two ships per day. The harbor only has one quay where a ship may dock to unload its cargo. If the queue is vacant at the arrival instant of a ship, the ship will dock to the quay for the duration of unloading its cargo. Otherwise an arriving ship will anchor at a harbor to wait for the quay to become vacant. The unloading times of cargos are exponentially distributed, with mean 3 hours, and independent of each other and independent of the arrival times of the ships.

Unloading a cargo costs 2000 eur/hour. In addition, every ship that is docked to the quay or anchored at the harbor incurs costs at the rate of 100 eur/hour.

(a) Model the number of ships docked to the quay and anchored at the harbor as a continuous-time Markov chain (X_t) and draw a transition diagram for the the chain. (2 p).

(b) Find out the equilibrium distribution of the process (X_t). (1 p)

(c) What is the probability that the quay is vacant in equilibrium? (1 p)

(d) Find out the long-run aggregate cost rate of the harbor operations. (2 p)