FINAL EXAM, DIFFERENTIAL AND INTEGRAL CALCULUS I, MS-A0111

- Time: 31.5.2019, 13:00-16:00
- Equipment: None
- Answer each problem on a separate page. Each problem is worth 4 points.
- Motivate all solutions carefully. Answers without motivation give no points.
- Good luck, have fun! /Ragnar

Problem 1

Compute the limit

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3},$$

or show that the limit does not exist.

Problem 2

The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x^2 + ax + b & \text{if } x \geq 0 \end{cases},$$

for some constants $a, b \in \mathbb{R}$. We know that f is differentiable on all of $\mathbb{R}$. What is $f(1)$?

Problem 3

Sketch the graph of

$$f(x) = \frac{\ln x}{\sqrt{x}} \text{ on } (0, \infty].$$

Problem 4

Compute

$$\int_0^1 \arccos x \, dx.$$

Problem 5

An island has been invaded by minks. A group of researchers predict that the size of the mink population $y(t)$ after t years can be described by the equation

$$y'(t) = ay(t)(1000 - y(t))$$

for some constant $a \in \mathbb{R}$. They visit the island on the same date on two consecutive years, and find 100 minks the first year and 250 minks the second year. How many minks should they expect to find when they return exactly one year later?

Standard limits

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0}\frac{\ln(x+1)}{x} = 1$$
$$\lim_{x\to 0}\frac{e^x-1}{x} = 1$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$
$$\lim_{n\to\infty}\frac{\ln n}{n^\alpha} = 0 \text{ if } \alpha > 0$$
$$\lim_{n\to\infty}\frac{n^\alpha}{\beta^n} = 0 \text{ if } \beta > 1$$
$$\lim_{n\to\infty}\frac{\beta^n}{n!} = 0$$
$$\lim_{n\to\infty}\frac{n!}{n^n} = 0$$

Standard derivatives

$f(x)$	$f'(x)$
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos(x)^2}$
e^x	e^x
$\ln\|x\|$	$\frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{1}{-\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$g(h(x))$	$g'(h(x))h'(x)$
$g(x)h(x)$	$g(x)h'(x)+g'(x)h(x)$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$

Inverse functions

$$\arcsin : [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$$
$$\arccos : [-1,1] \to [0,\pi]$$
$$\arctan : \mathbb{R} \to [-\frac{\pi}{2},\frac{\pi}{2}]$$
$$\ln : (0,\infty) \to \mathbb{R}$$
$$\sqrt{\cdot} : [0,\infty) \to [0,\infty)$$

Trigonometric identities

$$\tan(x) = \frac{\sin x}{\cos x}$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\sin(x+y) = \sin x\cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x \sin y$$
$$\sin(x)^2 + \cos(x)^2 = 1$$

θ	radians	sin	cos	tan
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	$-$

Algebraic identities

$$a^{x+y} = a^x a^y$$
$$\ln(xy) = \ln x + \ln y$$
$$x^n - 1 = (x-1)(1+x+\cdots+x^{n-1})$$
$$(x+y)^2 = x^2+2xy+y^2$$

Newton Raphson update rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Taylor series

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad \text{if } -1 < x < 1$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{for all } x$$