TU-E2210 First midterm exam 23.10.2019

1. Explain with couple of sentences:

a) Brownian motion
b) Markov chain
c) Time value of option
d) (can't remember)
e) American option
f) Martingales

2.

a) Derive the formula:

3.3 Derivation of the stock price process

Let us show how the stock price S_t at time t:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma X_t}$$

is obtained from the price process

$$dS_t = \mu S_t dt + \sigma S_t dX_t,$$

given the initial stock price S_0

b) Explain Newton method. 1p from explanation, 1p from formula, 1p from picture

3.

a) 2p, right from exercise:

3. Show that

$$\int_0^t X_s^3 dX_s = \frac{1}{4}X_t^4 - \frac{3}{2}\int_0^t X_s^2 ds.$$

b) From same exercise. 4p. Instead of S^p_t, it was asked S^5_t

4. Let $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma X_t}$ be geometric Brownian motion. Let p be a positive constant. Compute $d(S_t^p)$, the differential of S_t raised to the power p.

solution: *We write $dS_t = \mu S_t dt + \sigma S_t dX_t$. For geometric Brownian motion S_t, the quadratic variation is $(dS_t)^2 = \sigma^2 S_t^2 dt$. Without loss of generality, we assume that $p \neq 1$. We write $f = S_t^p$ and apply Ito's lemma (2). Since $\partial f/\partial(S^p) = pS_t^{p-1}$, $\partial^2 f/\partial(S^p)^2 = p(p-1)S_t^{p-2}$, substituting these derivatives to the Ito's lemma give*

$$\begin{aligned} d(S_t^p) &= pS_t^{p-1}dS_t + \frac{1}{2}p(p-1)S_t^{p-2}(dS_t)^2 \\ &= pS_t^{p-1}(\mu S_t dt + \sigma S_t dX_t) + \frac{1}{2}p(p-1)S_t^{p-2}\sigma^2 S_t^2 dt \\ &= S_t^p[p\mu dt + p\sigma dX_t + \frac{1}{2}p(p-1)\sigma^2 dt] \\ &= S_t^p p[(\mu + \frac{p-1}{2}\sigma^2)dt + \sigma dX_t]. \end{aligned}$$

4. a) Explain put-call parity 2p

b) 4p, as this exercise but slightly different values.

Bid and ask prices

The *bid* is the price at which you can sell a certain asset and the ask price is the price at which you can buy it. The *ask* is higher than the bid, and the amount by which the ask exceeds the bid is called the bid-ask spread.

The bid and ask prices for a six months European call option with strike $E_c = 40$ on a non-dividend-paying stock with spot price $S_0 = 42$ (actual price) are 5 € and 5.5 €, respectively. The bid and ask prices for a six months European put with strike $E_P = 40$ on the same underlying asset are 2.75 € and 3.25 €, respectively. Assume that the risk-free interest rate is $r = 0$. Is there an arbitrage opportunity present?

5. This exercise with different values. It was asked to find probability that the put will expire in the money.

Consider a put option with strike $E = 55$ and maturity $T = 0.25$ years on a non-dividend paying asset with spot price $S_0 = 60$. The underlying asset follows a lognormal model with drift equal to the riskless interest rate $r = 0.05$ and volatility $\sigma = 0.3$.

(a) Find the probability that the put will expire in the money.

(b) compute $\mathcal{N}(-d_2)$, where in the Black-Scholes formula $d_2 = (log(S_0/E) + (r - .5\sigma^2)T)/(\sigma\sqrt{T})$.

You will need the following information: the scaling property of Brownian motion states that $X_T \stackrel{d}{=} \sqrt{T}X_1$ we so can write

$$\begin{aligned} S_t &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma X_T} \\ &\stackrel{d}{=} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}X_1}, \end{aligned}$$

where $X_1 \sim \mathcal{N}(0, 1)$ is a standard normal variable. Dividing both sides by S_0 and then taking the natural logarithm gives

6. Short essay about stochastic volatilities