1. For each of the following statements, state if it is correct or not. *Justify* your answer.

   (a) The following equations have all their roots inside the unit disc:

   (i) $z^2 - 1.5z + 0.9 = 0$ [2p]

   (ii) $z^3 - 2z^2 + 2z - 0.5 = 0$ [3p]

   (b) Consider the system with the following characteristic equation:

$$\chi(z) = z^2 - \beta z - 0.5, \quad \beta \geq 0.$$

   The system is stable for $0 \leq \beta < 0.5$. [2p]

   (c) Consider the following system

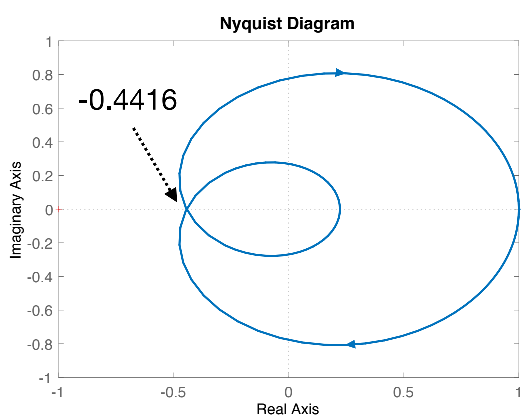
$$\begin{aligned} \mathbf{x}[k+1] &= \Phi \mathbf{x}[k] + \Gamma u[k] \\ y[k] &= C\mathbf{x}[k] \end{aligned}$$

   with

$$\Phi = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix}, \Gamma = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -4 \end{bmatrix}.$$

   The system is observable and reachable. [2p]

   (d) The Nyquist plot for $H(z) = \frac{0.4}{(z-0.2)(z-0.5)}$ is:



   The value of the gain K for which the system is stable is $K > 1/0.4416$. [2p]

   (e) When discretizing a continuous-time system with poles $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $|\lambda_{\max}| \triangleq \max_i |\lambda_i|$, using the state-space representation with sampling h and zero-order hold (ZOH), then the stability of the analog system is preserved and if $h < \pi/|\lambda_{\max}|$ there is no aliasing. [2p]

   (f) A discrete-time LTI system is reachable if it is possible to find a control sequence such that any state can be reached from any initial state in finite time. [2p]

(Turn over)

Solution.

(a) (i) **True**. This can be shown with 3 different ways:
1st way: Using the quadratic formula.

$$z_{1,2} = \frac{1.5 \pm \sqrt{(-1.5)^2 - 4(0.9)}}{2} = 0.75 \pm \frac{\sqrt{-1.35}}{2} = 0.75 \pm 0.58i$$
$$\Rightarrow |z_{1,2}|^2 = 0.75^2 + \frac{1.35}{4} = 0.9 < 1 \Rightarrow |z_{1,2}| < 1.$$

2nd way: Checking if the conditions of the triangle rule hold.

* $a_2 = 0.9 < 1$ ✓
* $a_2 = 0.9 > 1.5 - 1 = 0.5 = -a_1 - 1$ ✓
* $a_2 = 0.9 > -1.5 - 1 = -2.5 = a_1 - 1$ ✓

3rd way: Using Jury's stability criterion.

$\boxed{1}$	-1.5	0.9	
0.9	-1.5	1	$b_n = \frac{0.9}{1} = 0.9$
$\boxed{0.19}$	-0.15		
-0.15	0.19		$b_{n-1} = \frac{-0.15}{0.19} = -0.79$
$\boxed{0.07}$			

(ii) **False**. The most convenient way is to use the Jury's stability criterion.

$\boxed{1}$	-2	2	-0.5	
-0.5	2	-2	1	$b_n = \frac{-0.5}{1} = -0.5$
$\boxed{0.75}$	-1	1		
1	-1	0.75		$b_{n-1} = \frac{1}{0.75} = 1.3333$
$\boxed{-0.58}$	0.33			
$0,33$	-0.58			$b_{n-2} = \frac{0.33}{-0.58} = -0.57$
$\boxed{-0.39}$				

(b) **True**. There are 2 ways to do this.

1st way: Using the triangle rule.

- $-0.5 < 1$ ✓
- $-0.5 > \beta - 1 \Rightarrow \beta < 0.5$
- $-0.5 > -\beta - 1 \Rightarrow \beta > -0.5$, which holds anyway since $\beta \geq 0$.

2nd way: Using the quadratic formula.

$$z_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4(-0.5)}}{2} = \frac{\beta \pm \sqrt{\beta^2 + 2}}{2}$$

Then, since the poles are both real, we have 2 cases: the smallest pole should be bigger than -1 and the biggest pole should be smaller than 1.

$$\frac{\beta - \sqrt{\beta^2 + 2}}{2} > -1 \Rightarrow \beta + 2 > \sqrt{\beta^2 + 2} \Rightarrow \beta > -\frac{1}{2}.$$

$$\frac{\beta + \sqrt{\beta^2 + 2}}{2} < 1 \Rightarrow 2 - \beta > \sqrt{\beta^2 + 2}.$$

To be able to proceed with this inequality and given that the RHS is positive, we need that $2 - \beta > 0$, i.e., $\beta < 2$. Given that $\beta < 2$, we get

$$(2 - \beta)^2 > \beta^2 + 2 \Rightarrow \beta < 0.5.$$

Therefore, we need $0 \leq \beta < 0.5$.

(c) **False**. The system is reachable, but not observable.

- The controllability matrix is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}.$$

  $\det(W_c) = 2 \neq 0$ and, hence, the system is reachable.
- The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

  $\det(W_o) = 0$ and, hence, the system is not observable.

(d) **False**. There are 2 ways given for this.

1st way: The open loop system is stable ($p_1 = 0.2$ and $p_2 = 0.5$). Thus the closed loop system is stable if the Nyquist plot does not encircle the point -1. From the plot we see that:

$$K(-0.4416) > -1 \Rightarrow K < \frac{1}{0.4416}.$$

So, the statement is false.

2nd way: Using the triangle rule. The closed-loop transfer function $T(z)$ is given by

$$T(z) = \frac{KH(z)}{1 + KH(z)} = \frac{0.4K}{(z - 0.2)(z - 0.5) + 0.4K}.$$

The characteristic equation is therefore:

$$\chi(z) = (z - 0.2)(z - 0.5) + 0.4K = z^2 \underbrace{-0.7}_{a_1} z + \underbrace{0.1 + 0.4K}_{a_2}.$$

Using the triangle rule.

- $-1 < 0.1 + 0.4K < 1 \Rightarrow -2.75 < K < 2.25$
- $-0.7 - 1 < 0.1 + 0.4K \Rightarrow -4.5 < K$
- $0.7 - 1 < 0.1 + 0.4K \Rightarrow -1 < K$.

Therefore, the system is stable for $-1 < K < 2.25$; so, the statement is false.

(e) **True**. The poles of continuous-time systems are mapped to discrete through

$$p_i = e^{\lambda_i h}, \text{ where } \lambda_i = \sigma_i + j\omega_i$$

Therefore,

$$p_i = e^{\lambda_i h} = e^{(\sigma_i + j\omega_i)h} = e^{\sigma_i h} e^{j\omega_i h}$$

Regarding stability,

$$|p_i| = |e^{\lambda_i h}| = |e^{\sigma_i h} e^{j\omega_i h}| = e^{\sigma_i h} < 1, \text{ if and only if } \sigma_i < 0.$$

Regarding aliasing, from the Nyquist criterion, there is no aliasing if

$$\omega_s = \frac{2\pi}{h} > 2\omega_0 \Rightarrow h < \frac{\pi}{\omega_0}.$$

If the imaginary part of a pole of the continuous-time system is bigger than π/h then the frequency response has a peak at a higher frequency than the cut-off frequency ω_0 in the discrete-time domain, i.e.,

$$\omega_i h < \pi \Rightarrow \omega_i \leq \omega_0.$$

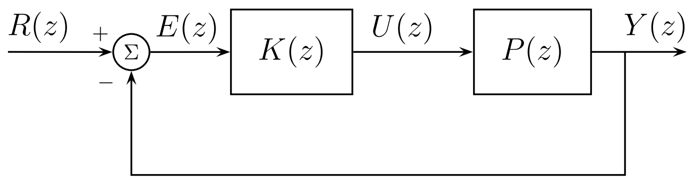
Therefore, since

$$|\lambda_{\max}| = |\sigma_{\max} + j\omega_{\max}| \geq |\omega_{\max}|,$$

then if $|\lambda_{\max}| \leq \omega_0$, then there will be no aliasing.

(f) **True**. By definition.

2. Consider the feedback system



where

$$P(z) = \frac{1}{z^2 + z + 0.9}$$

and K is a constant.

a) Draw the pole/zero diagram (z-plane) for the *open-loop* system $P(z)$. Is the system stable? [3p]

b) Find the closed-loop transfer function from $R(z)$ to $Y(z)$ as a function of $K(z)$. [3p]

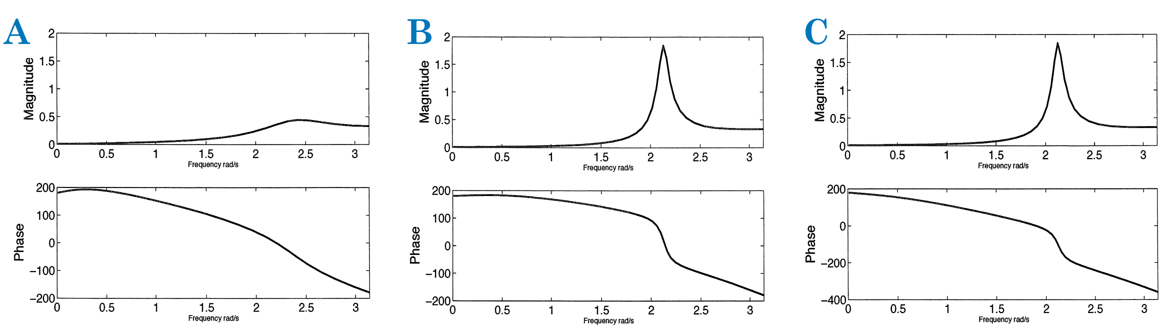
c) For which values of $K(z) = K$ (K is a constant value) is the closed-loop stable? [3p]

d) Consider the closed-loop system and let the input $r[k]$ be a unit step. Find, as a function of gain $K(z) = K$, the steady-state value of $y[k]$ (i.e., the $\lim_{k\to\infty} y[k]$) when this is finite, stating for which values of K the answer is valid. [3p]

e) Let

$$K = -\frac{1}{10}\frac{z-0.5}{z+0.5}.$$

The figure below shows three Bode plots (A, B and C), but only one corresponds to $K(z)P(z)$.



Choose the correct one, justifying your answer. [3p]

Solution.

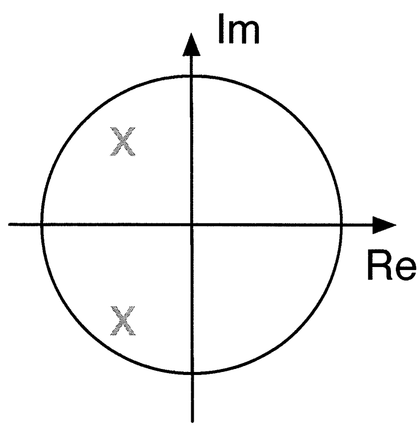
a) The open loop system has two poles:

$$p_{12} = \frac{-1 \pm \sqrt{1 - 4(0.9)}}{2} = \frac{-1 \pm \sqrt{-2.6}}{2} = -0.5 \pm j0.8062.$$

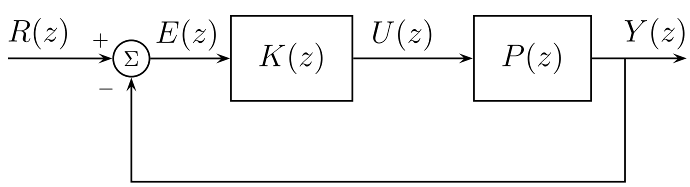
The magnitude of the poles is

$$|p_{12}| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{-2.6}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{2.6}{4}} = \sqrt{\frac{3.6}{4}} = \sqrt{0.9} < 1.$$

Hence, the poles are within the unit circle.



b) From the block diagram:



we have that

$$\begin{aligned} E(z) &= R(z) - Y(z) \\ &= R(z) - \underbrace{P(z)U(z)}_{Y(z)} \\ &= R(z) - P(z)\underbrace{K(z)E(z)}_{U(z)}. \end{aligned}$$

Therefore,

$$E(z) = \frac{R(z)}{1 + P(z)K(z)}.$$

Multiplying both sides with $P(z)K(z)$ we get

$$\underbrace{P(z)K(z)E(z)}_{Y(z)} = \frac{P(z)K(z)R(z)}{1+P(z)K(z)} \Rightarrow H(z) \triangleq \frac{Y(z)}{R(z)} = \frac{P(z)K(z)}{1+P(z)K(z)}$$

Therefore, the closed loop transfer function is given by

$$H(z) = \frac{\frac{K(z)}{z^2+z+0.9}}{1+\frac{K(z)}{z^2+z+0.9}} = \frac{K(z)}{z^2+z+0.9+K(z)}$$

c) The closed loop poles are the roots of $z^2+z+0.9+K=0$. Therefore, invoking the triangle rule we get:

$$\begin{cases} 0.9+K<1 \Rightarrow K<0.1 \\ 0.9+K>-1-1 \Rightarrow K>-2.9 \\ 0.9+K>1-1 \Rightarrow K>-0.9 \end{cases}$$

Combining all, we get $-0.9<K<0.1$.

d) When $K \notin (-0.9, 0.1)$ the system is unstable and, hence, $y[k]$ will grow unbounded. When $K \in (-0.9, 0.1)$, the closed loop system is stable and we use the final value theorem (using the closed loop transfer function $H(z)$ we found in part (b)):

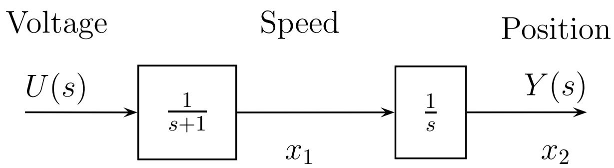
$$\begin{aligned} y_\infty &\triangleq \lim_{k\to\infty} y[k] = \lim_{z\to 1}(z-1)H(z)U(z) \\ &= \lim_{z\to 1}(z-1)\frac{K(z)}{z^2+z+0.9+K(z)}\frac{z}{z-1} \\ &= \frac{K}{2.9+K} \end{aligned}$$

e) – At $z=-1$, $P(1)K(1) = -0.0115$ and therefore the phase must be -180^o. Therefore, C cannot be the one.
   – Evaluating $P(z)K(z)$ at $z=e^{j2.1}$, we get $|P(e^{j2.1})K(e^{j2.1})| = 1.6$. Therefore, A cannot be the one.

Thus the correct one is B.

(Turn over)

3. A DC motor can be described by a second-order model with one time constant and one integrator; a normalized model of the motor is depicted in the simple block diagram below. Input $U(s)$ is the input voltage and output $Y(s)$ the shaft position.



a) Show that the state-space representation of the system (by using the state variables x_1 and x_2 in the figure) is given by [3p]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

b) Sample the state-space model with sampling time h, assuming ZOH and determine the discrete state-space representation of the form: [3p]

$$\mathbf{x}(kh+h) = \Phi(h)\mathbf{x}(kh) + \Gamma(h)\mathbf{u}(kh)$$
$$\mathbf{y}(kh) = C\mathbf{x}(kh) + D\mathbf{u}(kh)$$

c) Find the pulse transfer function of the discrete-time representation. [3p]

d) Determine the deadbeat controller of the motor. [3p]

e) Assume that $\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Determine the sample interval such that the control signal $\mathbf{u}(kh)$ is less than 1 in magnitude. It can be assumed that the maximum value of $\mathbf{u}(kh)$ is at $k = 0$. [3p]

Solution.

a) *1st way:* From the block diagram, it is clear that $y(t) = x_2(t)$ and since the signal $x_1(t)$ passes through the integrator block $(1/s)$ we conclude that $x_2(t) = \dot{x}_1(t)$. Hence, we have

$$y(t) = x_2(t) \Rightarrow Y(s) = X_2(s),$$
$$\dot{y}(t) = \dot{x}_2(t) = x_1(t) \Rightarrow sY(s) = sX_2(s) = X_1(s),$$
$$\ddot{y}(t) = \dot{x}_1(t) \Rightarrow s^2 Y(s) = sX_1(s)$$

The transfer function between the input and output is found by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}\frac{1}{s} = \frac{1}{s^2+s}, \tag{1}$$

so we have

$$s^2Y(s) + sY(s) = U(s) \Rightarrow sX_1(s) + X_1(s) = U(s) \Rightarrow \dot{x}_1(t) = -x_1(t) + u(t).$$

Hence, the state-space representation of the system with states $x_1(t)$ and $x_2(t)$ can be described as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

2nd way: Define the state vector as $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$. From the diagram, it is easily shown that $y(t) = x_2(t)$. Hence, $y(t) = \begin{bmatrix}0 & 1\end{bmatrix} x$. The transfer function between the input and output is found by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}\frac{1}{s} = \frac{1}{s^2+s}. \tag{2}$$

The system can be written in controllable canonical form from the TF with coefficients: $b_1 = 0$, $b_2 = 1$, $a_1 = 1$, $a_2 = 0$. Therefore,

$$\dot{x}(t) = \begin{bmatrix}-a_1 & -a_2\\ 1 & 0\end{bmatrix}x(t) + \begin{bmatrix}1\\ 0\end{bmatrix}u(t) = \begin{bmatrix}-1 & -0\\ 1 & 0\end{bmatrix}x(t) + \begin{bmatrix}1\\ 0\end{bmatrix}u(t),$$

which is the given system.

b) Assuming ZOH and sampling time h, matrices $\Phi(h)$ and $\Gamma(h)$ are found to be

$$\Phi(h) = e^{Ah} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}|_{t=h} = \mathcal{L}^{-1}\left\{\begin{bmatrix}\frac{1}{s+1} & 0\\ \frac{1}{s(s+1)} & \frac{1}{s}\end{bmatrix}\right\}\Bigg|_{t=h} = \begin{bmatrix}e^{-h} & 0\\ 1-e^{-h} & 1\end{bmatrix}$$
$$\Gamma(h) = \int_0^h e^{As}ds B = \int_0^h \begin{bmatrix}e^{-s} & 0\\ 1-e^{-s} & 1\end{bmatrix} ds \begin{bmatrix}1\\ 0\end{bmatrix} = \int_0^h \begin{bmatrix}e^{-s}\\ 1-e^{-s}\end{bmatrix} ds = \begin{bmatrix}1-e^{-h}\\ h+e^{-h}-1\end{bmatrix}$$

c) The discrete-time transfer function from discrete-time state-space representation is given by

$$\begin{aligned}
G(z) &= C(zI-\Phi)^{-1}\Gamma + D\\
&= \begin{bmatrix}0 & 1\end{bmatrix}\left(zI - \begin{bmatrix}e^{-h} & 0\\ 1-e^{-h} & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}1-e^{-h}\\ h+e^{-h}-1\end{bmatrix}\\
&= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}z-e^{-h} & 0\\ -1+e^{-h} & z-1\end{bmatrix}^{-1}\begin{bmatrix}1-e^{-h}\\ h+e^{-h}-1\end{bmatrix}\\
&= \frac{1}{(z-1)(z-e^{-h})}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}z-1 & 0\\ 1-e^{-h} & z-e^{-h}\end{bmatrix}\begin{bmatrix}1-e^{-h}\\ h+e^{-h}-1\end{bmatrix}\\
&= \frac{1}{(z-1)(z-e^{-h})}\begin{bmatrix}1-e^{-h} & z-e^{-h}\end{bmatrix}\begin{bmatrix}1-e^{-h}\\ h+e^{-h}-1\end{bmatrix}\\
&= \frac{(1-e^{-h})^2 + (z-e^{-h})(h+e^{-h}-1)}{(z-1)(z-e^{-h})}
\end{aligned}$$

(Turn over)

d) The system under consideration:

$$\begin{aligned}
\mathbf{x}[k+1] &= \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} \mathbf{x}[k] - \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} \mathbf{x}[k] \\
&= \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} \mathbf{x}[k] - \begin{bmatrix} (1-e^{-h})l_1 & (1-e^{-h})l_2 \\ (h+e^{-h}-1)l_1 & (h+e^{-h}-1)l_2 \end{bmatrix} \mathbf{x}[k] \\
&= \begin{bmatrix} e^{-h}-(1-e^{-h})l_1 & -(1-e^{-h})l_2 \\ 1-e^{-h}-(h+e^{-h}-1)l_1 & 1-(h+e^{-h}-1)l_2 \end{bmatrix} \mathbf{x}[k]
\end{aligned}$$

The corresponding characteristic polynomial is:

$$\begin{aligned}
\chi(z) &= \det(zI - \Phi + \Gamma L) \\
&= \begin{vmatrix} z-e^{-h}+(1-e^{-h})l_1 & (1-e^{-h})l_2 \\ -1+e^{-h}+(h+e^{-h}-1)l_1 & z-1+(h+e^{-h}-1)l_2 \end{vmatrix}
\end{aligned}$$

Let $a := e^{-h}$ and $b := h + e^{-h} - 1$. Then, $\chi(z)$ can be simplified to

$$\begin{aligned}
\chi(z) &= \begin{vmatrix} z-a+(1-a)l_1 & (1-a)l_2 \\ -1+a+bl_1 & z-1+bl_2 \end{vmatrix} \\
&= (z-a+(1-a)l_1)(z-1+bl_2) - (1-a)l_2(-1+a+bl_1) \\
&= z^2 + z[(-1+bl_2) + (-a+(1-a)l_1)] \\
&\quad + (-a+(1-a)l_1)(-1+bl_2) - (1-a)l_2(-1+a+bl_1)
\end{aligned}$$

Since we want a deadbeat control, the determinant should be equal to z^2 (i.e., the poles are equal to zero). Therefore,

$$\begin{cases} (-a+(1-a)l_1)(-1+bl_2) - (1-a)l_2(-1+a+bl_1) = 0 \\ (-1+bl_2) + (-a+(1-a)l_1) = 0 \end{cases}$$

from which l_1 and l_2 can be extracted. After algebraic manipulation

$$\begin{cases} -(1-a)l_1 - abl_2 + (1-a)^2 l_2 = -a \\ (1-a)l_1 + bl_2 = 1+a \end{cases} \tag{3}$$

Adding the two equations we get

$$\begin{aligned}
bl_2 - abl_2 + (1-a)^2 l_2 = 1 &\Rightarrow (1-a)bl_2 + (1-a)^2 l_2 = 1 \\
&\Rightarrow (1-a)(b+1-a)l_2 = 1 \\
&\Rightarrow (1-a)hl_2 = 1 \qquad \text{(note: } b = h+a-1\text{)} \\
&\Rightarrow \boxed{l_2 = \frac{1}{h(1-e^{-h})}}
\end{aligned}$$

Substituting l_2 back in (3), we get

$$(1-a)l_1 + \frac{b}{h(1-a)} = 1+a \Rightarrow \boxed{l_1 = \frac{1}{1-a}\left(1+a-\frac{b}{h(1-a)}\right)}$$

e) In this case,

$$u[0] = \begin{bmatrix} l_1 & l_2 \end{bmatrix} \mathbf{x}[0] = l_1.$$

Since we want $u[0] < 1$ we should choose h such that $l_1 < 1$. Therefore, from part (b) we get

$$\begin{aligned}
\frac{1}{1-a}\left(1 + a - \frac{b}{h(1-a)}\right) < 1 &\Rightarrow 1 - a > 1 + a - \frac{b}{h(1-a)} \\
&\Rightarrow \frac{b}{h(1-a)} > 2a \\
&\Rightarrow h + e^{-h} - 1 > 2he^{-h}(1 - e^{-h})
\end{aligned}$$

An h should be chosen such that the inequality above is satisfied.