CS-E5795 Computational Methods in Stochastics
Exam 10.12.2019

1. a) (2 p) Describe at least three properties of a good random number generator.
   b) (2 p) Describe how **one** (pseudo)random number generator works. (One fundamental generator – no transformations etc. The operation principle suffices. Values of possible parameters are not important.)
   c) (2 p) In an assignment you fiddled around with logarithmic binning. What is the motivation for doing it? In what kind of situations would you try using it?

2. Explain the principles of and motivations for the following methods. You may use example distributions or probability density functions, if you like.
   a) (3 p) Importance sampling.
   b) (3 p) Rejection sampling.

3. a) (2 p) Your boss wants you to simulate a discrete stochastic process saying that he has worked out the Markov matrix for it and shows you this:

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0 & 0.1 & 0 \\ 0 & 0.2 & 0.2 & 0.5 & 0.4 & 0 \\ 0.5 & 0.1 & 0 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0.5 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}$$

   Will you write the algorithm and simulate? Justify your decision.

   b) (4 p) In simulating an inhomogeneous Poisson process, the thinning approach differs from most other approaches by sampling inter-event times. Write a pseudo algorithm to describe this simulation procedure.

4. a) (2 p) Gibbs and Metropolis-Hastings (M-H) algorithms. How (in which context) are they used? How do they differ; when would you use Gibbs and when M-H?
   b) (4 p) Describe in detail how you would simulate the distribution, whose probability density function (pdf) is $f(x) = 3\exp[-3(x+2)]$, where $x \in [-2, \infty)$, using the inverse distribution method. Check that you do get the support (range) $[-2, \infty)$ for the distribution with your procedure. $\rightarrow [3, 0)$

$f(x) = u = 3e^{-3(x+2)}$

Turn the page, please.

$\ln \frac{u}{3} = -3(x+2)$

$x = -2 - \frac{1}{3} \ln \frac{u}{3}$ $\qquad [3,0) \longrightarrow [-2, \infty)$

$\Rightarrow f^{-1}(x) = -2 - \frac{1}{3} \ln \frac{x}{3}$ $\qquad$ uniform sampling

5. a) (4 p) Describe the two steps comprising Hamiltonian Monte Carlo (HMC) method. When appropriate, explain reasons for the details in the procedure?
b) (2 p) The most straightforward case in using HMC is when the target distribution is Gaussian. Justify why using the potential energy term $U(q) = \frac{1}{2}q^2$ corresponds to simulating a Gaussian distribution. (You do not need to worry about the normalisation constant for the Gaussian pdf here.)

For reference:

Gaussian (normal) distribution:

The probability density function

$$\phi(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty.$$

Poisson distribution:

The probability mass function ($\lambda > 0$)

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad \text{for } k = 0,\ 1, \ldots$$