# ELEC-E7210 Communication Theory

**This is a closed book exam** All five tasks are evaluated and taken into account in the grading. The exam can be written in Finnish, Swedish or English.

## Exam 12.12. 2019

1. Consider a uniform linear antenna array with $N_t$ transmit antennas.

   a) Compute the array gain for $N_t = 2, 4, 6, 8$. What can you say about the benefit of expanding the array with additional antennas?

   b) What would be the benefit of exploiting the spatial diversity of the array?

   c) The receiver has $N_r$ antennas. What is the maximal multiplexing gain?

   d) How could you exploit the array in a multiuser scenario?

2. A Mobile Station is moving with a velocity of $v = 3$ km/h, and communicating on a carrier frequency of $f_c = 1$ GHz. The symbol duration is $T = 5$ $\mu$s and the bandwidth is $W = 1/T = 200$ kHz. The channel is a tapped delay line with three channel taps

| Tap $i$ | 1 | 2 | 3 |
|---|---|---|---|
| Delay $\tau_i$ [$\mu$s] | 0 | 0.2 | 0.5 |
| Power $P_i$ [dB] | -3 | 0 | -4 |

   a) What is the maximum Doppler shift and the Doppler spread of the channel? Estimate the coherence time. Is the signal at the mobile station rapidly or slowly fading?
   Hint: The speed of light is $c = 3 \cdot 10^8$m/s

   b) What is the maximum excess delay of the channel? Estimate the coherence bandwidth. Is the channel frequency selective or frequency flat?

3. A signal is received over two independently fading channels in signal space. The fading coefficients of the channels are $h_1 = 6$ and $h_2 = 2$, and the amplitude of the transmitted symbols is 1. In both channels the noise variance is the same, $\sigma_N^2 = 4$. We assume a real channel with real signals and real noise.
   a) Calculate the received SNR if the receiver uses only the first channel.
   b) Calculate the received SNR for maximum ratio combining (MRC).
   c) Calculate the received SNR for equal gain combining (EGC).
   d) Express the gain achieved while using MRC and EGC compared to using only the first channel as in a).

4. Consider the vector signal model $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$. The noise is white and Gaussian, and the variance of the noise samples is $\mathrm{E}\{n_i n_i^*\} = \mathrm{N}_0$. The expected symbol energy is one: $\mathrm{E}\{x_i x_i^*\} = 1$. The channel matrix is $\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

a) What is the post-processing signal-to-interference-and-noise ratio (SINR) experienced by symbols $x_1$ and $x_2$, after Zero Forcing processing has been used in the receiver to mitigate crosstalk? Hint: A Zero-Forcing receiver removes all interference, but amplifies noise. Noise variance is amplifed by a factor given by the diagonal elements of the inverse of the channel covariance matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$. Note that the noise affecting the transmitted signals $x_1$ and $x_2$ is amplified with a different factor.

b) What would the SINRs be if all crosstalk (off-diagonal terms in $\mathbf{H}$) would be removed?

5. Consider a static $r$-Tx and $r$-Rx antenna MIMO channel, with $r \times r$ channel matrix $\mathbf{H}$, known perfectly to Tx and Rx. Given codeword $\mathbf{x} = [x_1, x_2, \ldots, x_r]^T$, the Rx signal is modeled as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n},$$

where the vector $\mathbf{n}$ is white Gaussian noise. Given matrix $\mathbf{H}$, the capacity of the channel is

$$C_{\mathbf{H}}(\rho) = B \max_{\rho_i:\, \Sigma_i \rho_i \leq \rho} \sum_i \log_2(1 + \lambda_i \rho_i),$$

where $\rho$ is the average transmit SNR, i.e. the ratio of the transmit power to receiver noise power, $B$ is bandwidth, and $\sqrt{\lambda_i}$ are singular values of $\mathbf{H}$. Now take $B = 100$ KHz, $\rho = 10$ dB, and $\mathbf{H} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) Find the singular values of $\mathbf{H}$. Express the MIMO signal model with this channel matrix in an equivalent form in terms of parallel channels. Hint: You find the singular values by first finding the eigenvalues $\lambda$ by solving the equation $\det\left(\mathbf{H}^{\mathrm{T}}\mathbf{H} - \lambda\mathbf{I}\right) = 0$, and taking the square root of the eigenvalues.

b) Find the capacity of the channel with this channel matrix.

Hint: Note that

$$\log_2(1+x) + \log_2(1+y) \leq 2\log_2((1+y+1+x)/2)$$

for all positive $x$ and $y$.