Questions of the exam CONVEX OPTIMIZATION II (2019) by Prof Sergiy Vorobyov:

(below are questions asked during the exam, the QP was taken back after exam)

**5.27** *Equality constrained least-squares.* Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m \times n}$ with $\mathbf{rank}\, A = n$, and $G \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, G = p$.
Give the KKT conditions, and derive expressions for the primal solution $x^\star$ and the dual solution $\nu^\star$.

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*Composition of linear-fractional functions.* Suppose $\phi : \mathbf{R}^n \to \mathbf{R}^m$ and $\psi : \mathbf{R}^m \to \mathbf{R}^p$ are the linear-fractional functions

$$\phi(x) = \frac{Ax + b}{c^T x + d}, \qquad \psi(y) = \frac{Ey + f}{g^T y + h},$$

with domains $\mathbf{dom}\, \phi = \{x \mid c^T x + d > 0\}$, $\mathbf{dom}\, \psi = \{y \mid g^T x + h > 0\}$. We associate with $\phi$ and $\psi$ the matrices

$$\begin{bmatrix} A & b \\ c^T & d \end{bmatrix}, \qquad \begin{bmatrix} E & f \\ g^T & h \end{bmatrix},$$

respectively.
Now consider the composition $\Gamma$ of $\psi$ and $\phi$, *i.e.*, $\Gamma(x) = \psi(\phi(x))$, with domain

$$\mathbf{dom}\, \Gamma = \{x \in \mathbf{dom}\, \phi \mid \phi(x) \in \mathbf{dom}\, \psi\}.$$

Show that $\Gamma$ is linear-fractional, and that the matrix associated with it is the product

$$\begin{bmatrix} E & f \\ g^T & h \end{bmatrix} \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}.$$

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**12.12** *Spectral factorization via semidefinite programming.* A Toeplitz matrix is a matrix that has constant values on its diagonals. We use the notation

$$T_m(x_1, \ldots, x_m) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{m-1} & x_m \\ x_2 & x_1 & x_2 & \cdots & x_{m-2} & x_{m-1} \\ x_3 & x_2 & x_1 & \cdots & x_{m-3} & x_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1} & x_{m-2} & x_{m-2} & \cdots & x_1 & x_2 \\ x_m & x_{m-1} & x_{m-2} & \cdots & x_2 & x_1 \end{bmatrix}$$

to denote the symmetric Toeplitz matrix in $\mathbf{S}^{m\times m}$ constructed from $x_1, \ldots, x_m$. Consider the semidefinite program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & T_n(x_1, \ldots, x_n) \succeq e_1 e_1^T, \end{array}$$

with variable $x = (x_1, \ldots, x_n)$, where $e_1 = (1, 0, \ldots, 0)$.

(a) Derive the dual of the SDP above. Denote the dual variable as $Z$. (Hence $Z \in \mathbf{S}^n$ and the dual constraints include an inequality $Z \succeq 0$.)

(b) Show that $T_n(x_1, \ldots, x_n) \succ 0$ for every feasible $x$ in the SDP above. You can do this by induction on $n$.

- For $n = 1$, the constraint is $x_1 \geq 1$ which obviously implies $x_1 > 0$.
- In the induction step, assume $n \geq 2$ and that $T_{n-1}(x_1, \ldots, x_{n-1}) \succ 0$. Use a Schur complement argument and the Toeplitz structure of $T_n$ to show that $T_n(x_1, \ldots, x_n) \succeq e_1 e_1^T$ implies $T_n(x_1, \ldots, x_n) \succ 0$.

(c) Suppose the optimal value of the SDP above is finite and attained, and that $Z$ is dual optimal. Use the result of part (b) to show that the rank of $Z$ is at most one, *i.e.*, $Z$ can be expressed as $Z = yy^T$ for some $n$-vector $y$. Show that $y$ satisfies

$$\begin{aligned} y_1^2 + y_2^2 + \cdots + y_n^2 &= c_1 \\ y_1 y_2 + y_2 y_3 + \cdots + y_{n-1} y_n &= c_2/2 \\ &\vdots \\ y_1 y_{n-1} + y_2 y_n &= c_{n-1}/2 \\ y_1 y_n &= c_n/2. \end{aligned}$$



This can be expressed as an identity $|Y(\omega)|^2 = R(\omega)$ between two functions

$$\begin{aligned} Y(\omega) &= y_1 + y_2 e^{-i\omega} + y_3 e^{-3i\omega} + \cdots + y_n e^{-i(n-1)\omega} \\ R(\omega) &= c_1 + c_2 \cos\omega + c_3 \cos(2\omega) + \cdots + c_n \cos((n-1)\omega) \end{aligned}$$

(with $i = \sqrt{-1}$). The function $Y(\omega)$ is called a *spectral factor* of the trigonometric polynomial $R(\omega)$.

**Problem 6:** (5 points) *Gradient and Newton methods for composition functions.* Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is increasing and convex, and $f : \mathbf{R}^n \to \mathbf{R}$ is convex, so $g(x) = \phi(f(x))$ is convex. (We assume that $f$ and $g$ are twice differentiable.) The problems of minimizing $f$ and minimizing $g$ are clearly equivalent.

Compare the gradient method and Newton's method, applied to $f$ and $g$. How are the search directions related? How are the methods related if an exact line search is used?

*Hint.* Use the matrix inversion lemma.