(a) (3 points) Write down the left and rights cosets for $K = \{I, F_1\}$. Is K a normal subgroup of D_3? Justify your answer. no

(b) (3 points) Let $H = \{I, R, R^2\}$. Then H is a normal subgroup of D_3. Write down the Cayley table for the quotient group D_3/H.

(c) (4 points) Let R be a group and $f : D_3 \to R$ be a group homomorphism. What are the possible images $f(D_3)$ of D_3 under the group homomorphism f (up to isomorphism)? Justify your answer. Hint: Use the (first) isomorphism theorem for groups.

3. (a) (5 points) Let $\mathbb{R}[x]$ be the ring of real polynomials with standard addition and multiplication. Which of the following subsets of $\mathbb{R}[x]$ are also subrings of $\mathbb{R}[x]$? Which of them are ideals?

   i. Constant polynomials (where $f(x) = c$ for some constant $c \in \mathbb{R}$).

   ii. Even polynomials (where $f(x) = f(-x)$).

   iii. Odd polynomials (where $f(-x) = -f(x)$).

   iv. Polynomials where $f(0) = 0$.

   v. Polynomials where $f(1) = 1$.

   (b) (5 points) Let R, R' be rings and $f : R \to R'$ be a ring homomorphism. Let I be an ideal of R. Prove that $f(I)$ is an ideal of $f(R)$.

4. (a) (5 points) Let G be a group and let $g \in G$ be an element of finite order m. Prove that for any integers i and j, $g^i = g^j$ if and only if m divides $i - j$.

   (b) (5 points) Let G, G' be groups and let $f : G \to G'$ be a group homomorphism. Let $g \in G$ be an element of finite order m. Prove that the order of $f(g)$ is a divisor of m.

5. (a) (4 points) List all subgroups of the group $(\mathbb{Z}_6, +)$. Justify why the list of subgroups is complete.

   (b) (2 points) Determine the order of each element of $(\mathbb{Z}_6, +)$ and list the generators of $(\mathbb{Z}_6, +)$.

   (c) (4 points) Give all group homomorphisms from $(\mathbb{Z}_8, +)$ to $(\mathbb{Z}_6, +)$. Justify your answer.

6. For any prime p, let $E_p = \{a + b\sqrt{p} : a, b \in \mathbb{Q}\}$, so that $E_p \subseteq \mathbb{R}$.

   (a) (5 points) Prove that E_p is a subfield of $\mathbb{R}$.

   (b) (5 points) Prove that E_2 and E_3 are not isomorphic.

Abstract Algebra Exam, MS-C1081

Department of Mathematics and Systems Analysis, Aalto University

20.02.2020

You may bring to the exam a memory aid sheet of size A4. The memory aid sheet must be hand-written, contain text on one side only, and have your name and student number written in the top right corner. You do not need to return the memory aid sheet.

- If you are taking both MS-C1081 and MS-C1540 course exams, then do problems 1-3. Your final points for the course will be 5/3*(exam points) + exercise points. You may choose to do either of the options below, in which case your final points for the course will be maximum over all the options.
- If you are taking MS-C1081 course exam (KT), then do problems 1-5. Your final points for the course will be exam points + exercise points. You may choose to do the option below, in which case your final points for the course will be maximum of the two options.
- If you are taking MS-C1081 general exam (T0), then do problems 1-6. Your final points for the course will be 5/3*(exam points).

Problems:

1. (a) (4 points) Does the set $\{1, -1, i, -i\}$ with the binary operation given by the multiplication of complex numbers form a group? Justify your answer.
   (b) (3 points) Give an example of a non-abelian group and justify why it is non-abelian.
   (c) (3 points) Let $(G, +)$ be an abelian group and define $H = \{a \in G : 4a = a\}$. Show that H is a subgroup of G.

2. Consider the dihedral group $D_3 = \{I, R, R^2, F_1, F_2, F_3\}$. Let us recall that D_3 is the symmetry group of an equilateral triangle and its Cayley table is given by

$\cdot$	I	R	R^2	F_1	F_2	F_3
I	I	R	R^2	F_1	F_2	F_3
R	R	R^2	I	F_3	F_1	F_2
R^2	R^2	I	R	F_2	F_3	F_1
F_1	F_1	F_2	F_3	I	R	R^2
F_2	F_2	F_3	F_1	R^2	I	R
F_3	F_3	F_1	F_2	R	R^2	I