Abstract Algebra Exam, MS-C1081

Department of Mathematics and Systems Analysis, Aalto University

08.06.2020

You may bring to the exam a memory aid sheet of size A4. The memory aid sheet must be hand-written, contain text on one side only, and have your name and student number written in the top right corner. You do not need to return the memory aid sheet.

- If you attended MS-C1081 in 2020, then you may choose to do problems 1-5. Your final points for the course will be exam points + exercise points. You may choose to do the option below, in which case your final points for the course will be maximum of the two options.
- Otherwise do problems 1-6. Your final points for the course will be 5/3*(exam points).

Problems:

1. (a) (5 points) Consider the set $\{e, a, b, c\}$ and binary operation $*$ on this set. Can the partial multiplication table for $*$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	
b	b		e	
c	c			e

be completed to the Cayley table of a group? If yes, how many ways are there to complete the table? Justify your answer.

(b) (3 points) Give an example of a non-abelian group and justify why it is non-abelian.

(c) (2 points) Let G be a group with binary operation $*$. Prove that $a * b = a * c$ implies $b = c$ for all $a, b, c \in G$.

(d) (2 points) Let G be a group of order 12 and let $g \in G$. What are the possible orders of the element g?

2. Recall that the elements of the symmetric group S_n are permutations on n elements and the binary operation is the composition of permutations.

(a) (1 point) Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

in the symmetric group S_6. Write σ in the cycle notation.

(b) (2 points) What is the order of σ in part (a)?

(c) (5 points) Find all cyclic subgroups of the symmetric group S_3.

(d) (3 points) Find all subgroups of the symmetric group S_3. Justify why the list of subgroups is complete.

(e) (1 point) Which subgroup in part (d) is the alternating group A_3?

3. (a) (3 points) Give an example of a desired subgroup and group if possible. If impossible, say why it is impossible.

   i. A subgroup of an abelian group G whose left cosets and right cosets give different partitions of G

   ii. A subgroup of a group G whose left cosets give a partition of G into just one cell

   iii. A subgroup of a group of order 6 whose left cosets give a partition of the group into 6 cells

   (b) (1 point) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$ with componentwise addition. Consider the subgroup $H = \langle(0,1)\rangle$, the subgroup generated by $(0,1)$. Is H a normal subgroup of G? Justify your answer.

   (c) (1 point) Let H and G be as in part (b). List all cosets for H.

   (d) (2 points) Write down the Cayley table for the quotient group G/H.

4. (a) (4 points) Decide whether the following operations of addition and multiplication are defined (closed) on the set, and give a ring structure. If a ring is formed, state whether the ring is commutative and whether it is a field. If a ring is not formed, tell why this is the case, otherwise no justifications are necessary.

   i. $n\mathbb{Z}$ with the usual addition and multiplication

   ii. $\mathbb{Z} \times \mathbb{Z}$ with addition and multiplication by components

   iii. $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ with the usual addition and multiplication

   iv. The set of all pure imaginary complex numbers ri for $r \in \mathbb{R}$ with the usual addition and multiplication

   (b) (2 points) Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$

   (c) (3 points) Describe all ring homomorphisms of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$.

5. (a) (5 points) Let R be a commutative ring and let $a_1, a_2, \ldots, a_k \in R$. Prove that the set

$$\{r_1 a_1 + r_2 a_2 + \ldots + r_k a_k : r_i \in R \text{ for } i \in \{1, 2, \ldots, k\}\}.$$

   is an ideal of R.

(b) (5 points) The ideal generated by the set $\{a_1, a_2, \ldots, a_k\}$ is denoted by $\langle a_1, a_2, \ldots, a_k\rangle$. Prove that

$$\langle a_1, a_2, \ldots, a_k\rangle = \{r_1 a_1 + r_2 a_2 + \ldots + r_k a_k : r_i \in R \text{ for } i \in \{1, 2, \ldots, k\}\}.$$

Hint: Use part (a) of this exercise.

6. (a) (4 points) Prove that the set $T = \{z \in \mathbb{C} : |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$.

   (b) (6 points) Show that the group T from part (a) is isomorphic to the quotient group $\mathbb{R}/\mathbb{Z}$.