1. Käyrä $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ kulkee pisteestä $(1,-1)$ pisteseen $(4,8)$ origon kautta. Laske käyrän pituus, sekä tarkka arvo (5p.) että pituus pyöristetty yhden desimaalin tarkuuteen (1p.).

Ratkaisu: Käyrällä $t \in [-1,2]$. $x(t) = t^2 \Rightarrow x'(t) = 2t$, $y(t) = t^3 \Rightarrow y'(t) = 3t^2$.
$\ell = \int_{-1}^{2} \sqrt{(x'(t))^2 + (y'(t))^2}dt = \int_{-1}^{2} \sqrt{4t^2 + 9t^4}dt = \int_{-1}^{2} |t|\sqrt{4+9t^2}dt = (\int_{-1}^{0}(-t)\sqrt{4+9t^2}dt) + (\int_{0}^{2} t\sqrt{4+9t^2}dt) = \{u = 4{+}9t^2 \Rightarrow du = 18tdt;\ t=-1 \Rightarrow u = 13, t = 0 \Rightarrow u = 4, t = 2 \Rightarrow u = 40\} = (-\int_{13}^{4}\sqrt{u}\frac{du}{18}) + (\int_{4}^{40}\sqrt{u}\frac{du}{18}) = ([\frac{1}{18}\cdot\frac{2}{3}u^{3/2}]_4^{13}) + ([\frac{1}{18}\cdot\frac{2}{3}u^{3/2}]_4^{40}) = \frac{1}{27}(13\sqrt{13} - 8 + 40\sqrt{40} - 8) = \frac{1}{27}(40\sqrt{40} + 13\sqrt{13} - 16) \approx 10.5$

2. a) Määritä funktion $g(x,y) = 3 + \ln(x^2 + \sin y)$ ensimmäisen asteen Taylorin polynomi $P_1(x,y)$ pisteen $(a,b) = (1,0)$ ympäristössä (2p.) ja approksimoi lukua $g(0.98, 0.03)$ luvulla $P_1(0.98, 0.03)$ (1p.).
b) Määritä funktion $g(x,y) = 3 + \ln(x^2 + \sin y)$ toisen asteen Taylorin polynomi $P_2(x,y)$ pisteen $(a,b) = (1,0)$ ympäristössä (2p.) ja approksimoi lukua $g(0.98, 0.03)$ luvulla $P_2(0.98, 0.03)$ (1p.).
(Mathematica: $g(0.98, 0.03) \approx 2.99034908$)

Ratkaisu: a) $g(a,b) = g(1,0) = 3 + \ln(1^2 + \sin 0) = 3$.
$\frac{\partial g}{\partial x} = \frac{2x}{x^2+\sin y} \Rightarrow \frac{\partial g}{\partial x}(1,0) = 2$, $\frac{\partial g}{\partial y} = \frac{\cos y}{x^2+\sin y} \Rightarrow \frac{\partial g}{\partial y}(1,0) = 1 \Rightarrow$
$P_1(x,y) = g(1,0) + \frac{\partial g}{\partial x}(1,0)\cdot(x-1) + \frac{\partial g}{\partial y}(1,0)\cdot(y-0) \Rightarrow P_1(0.98, 0.03) = 3 + 2\cdot(-0.02) + 1\cdot 0.03 = 2.99$.
b) $\frac{\partial^2 g}{\partial x^2} = \frac{2(x^2+\sin y) - 4x^2}{(x^2+\sin y)^2} \Rightarrow \frac{\partial^2 g}{\partial x^2}(1,0) = -2$, $\frac{\partial^2 g}{\partial y^2} = \frac{-\sin y(x^2+\sin y) - \cos^2 y}{(x^2+\sin y)^2} \Rightarrow \frac{\partial^2 g}{\partial y^2}(1,0) = -1$,
$\frac{\partial^2 g}{\partial y \partial x} = \frac{-2x\cos y}{(x^2+\sin y)^2} = \frac{\partial^2 g}{\partial x \partial y} \Rightarrow \frac{\partial^2 g}{\partial y \partial x}(1,0) = \frac{\partial^2 g}{\partial x \partial y}(1,0) = -2 \Rightarrow$
$P_2(x,y) = g(1,0) + (\frac{\partial g}{\partial x}(1,0)\cdot(x-1) + \frac{\partial g}{\partial y}(1,0)\cdot(y-0)) + \frac{1}{2!}(\frac{\partial^2 g}{\partial x^2}(1,0)\cdot(x-1)^2 + 2\frac{\partial^2 g}{\partial x \partial y}(1,0)\cdot(x-1)(y-0) + \frac{\partial^2 g}{\partial y^2}(1,0)\cdot(y-0)^2) \Rightarrow P_1(0.98, 0.03) = 2.99 + \frac{1}{2}(-2\cdot(-0.02)^2 + 2\cdot(-2)\cdot(-0.02)(0.03) + 1\cdot(0.03)^2) = 2.99 + \frac{1}{2}(-0.0008 + 0.0024 - 0.0009) = 2.99 + 0.00035 = 2.99035$.

3. Polynomi $p(x,y) = 3x^4 + 2y^6 - 12xy$ on määritelty koko xy-tasossa ja $p(x,y) \to \infty$, kun $\sqrt{x^2+y^2} \to \infty$, joten p:llä ei ole suurinta arvoa. Määritä p:n kriittiset pisteet ja pienin arvo. (6p.)

Ratkaisu: $\frac{\partial p}{\partial x} = 12x^3 - 12y = 0 \Rightarrow x^3 = y$, $\frac{\partial p}{\partial y} = 12y^5 - 12x = 0 \Rightarrow y^5 = x \Rightarrow y^{15} = x^3 = y \Rightarrow$
$(y = -1 \Rightarrow x = -1)$ tai $(y = 0 \Rightarrow x = 0)$ tai $(y = 1 \Rightarrow x = 1)$.
Yhteensä kolme kriittistä pistettä: $(-1,-1)$, $(0,0)$ ja $(1,1)$.
$p(0,0) = 0 > p(-1,-1) = p(1,1) = -7 = p_{min}$.

4. Käyrät $y = \cos(3x)$ ja $y = 3\cos(x)$ rajoittavat välillä $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ tasoalueen Ω, jonka pinta-ala on $A = \iint_\Omega dA$.
Ω:n keskiön x-koordinaatti $\bar{x} = \frac{1}{A}\cdot\iint_\Omega xdA = 0$ symmetrian takia. Määritä Ω:n pinta-ala A (1p.) ja sen keskiön y-koordinaatti $\bar{y} = \frac{1}{A}\cdot\iint_\Omega ydA$ (5p.).

Ratkaisu: $A = \int_{-\pi/2}^{\pi/2}(3\cos(x) - \cos(3x))dx = [3\sin(x) - \frac{1}{3}\sin(3x)]_{-\pi/2}^{\pi/2} = (3 + \frac{1}{3}) - (-3 - \frac{1}{3}) = 6\frac{2}{3} = \frac{20}{3}$.
$\bar{y} = \frac{1}{A}\cdot\int_{-\pi/2}^{\pi/2}(\int_{\cos(3x)}^{3\cos(x)} ydy)dx = \frac{1}{A}\cdot\int_{-\pi/2}^{\pi/2}[y^2/2]_{\cos(3x)}^{3\cos(x)}dx = \frac{1}{2A}\cdot\int_{-\pi/2}^{\pi/2}(9\cos^2(x) - \cos^2(3x))dx = \frac{1}{2A}\cdot\int_{-\pi/2}^{\pi/2}(\frac{9}{2}(1+\cos(2x)) - \frac{1}{2}(1+\cos(6x)))dx = \frac{1}{4A}\cdot[8x + \frac{9}{2}\sin(2x) - \frac{1}{6}\sin(6x)]_{-\pi/2}^{\pi/2} = \frac{1}{4A}\cdot 8\pi = \frac{3\pi}{10}$.

5. Pallon $B = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \le R^2\}$ tiheys pisteessä $(x,y,z) \in B$ on $\delta(x,y,z) = \delta_0\cdot(x^2+y^2)/R^2$, joten $\delta_{max} = \delta_0$ ("päiväntasaajalla" xy-tasossa) ja $\delta_{min} = 0\,[kg/m^3]$ (z-akselilla). Laske pallon massa. (6p.)

Ratkaisu: i) (sylinterikoordinaatit) $m = \iiint_B \delta dV = \int_0^{2\pi}(\int_{-R}^{R}(\int_0^{\sqrt{R^2-z^2}}(\delta_0\cdot r^2/R^2)\cdot rdr)dz)d\theta = \frac{\delta_0}{R^2}(\int_0^{2\pi}d\theta)\cdot(\int_{-R}^{R}[r^4/4]_0^{\sqrt{R^2-z^2}}dz) = \frac{2\pi\delta_0}{R^2}\int_{-R}^{R}\frac{1}{4}(R^4 - 2R^2z^2 + z^4)dz = \frac{\pi\delta_0}{2R^2}[R^4z - \frac{2}{3}R^2z^3 + \frac{1}{5}z^5]_{-R}^{R} = \frac{\pi\delta_0}{2R^2}\cdot 2(1 - \frac{2}{3} + \frac{1}{5})R^5 = \frac{8\pi}{15}\delta_0 R^3$.
ii) (pallokoordinaatit) $m = \iiint_B \delta dV = \int_0^{2\pi}(\int_0^{\pi}(\int_0^{R}(\delta_0\cdot(\rho\sin\phi)^2/R^2)\rho^2\sin\phi d\rho)d\phi)d\theta = \frac{\delta_0}{R^2}(\int_0^{2\pi}d\theta)(\int_0^{\pi}\sin^3\phi d\phi)(\int_0^R \rho^4 d\rho) = \frac{\delta_0}{R^2}\cdot 2\pi\cdot\int_0^{\pi}(1-\cos^2\phi)\sin\phi d\phi)\cdot[\rho^5/5]_0^R = \frac{\delta_0}{R^2}\cdot 2\pi\cdot[-\cos\phi + \frac{1}{3}\cos^3\phi]_0^{\pi}\cdot\frac{R^5}{5} = \frac{2\pi\delta_0}{R^2}\cdot\frac{4}{3}\cdot\frac{R^5}{5} = \frac{8\pi}{15}\delta_0 R^3$.
(Huom: Keskitiheys $\bar{\delta} = m/V = \frac{8\pi\delta_0 R^3/15}{4\pi R^3/3} = \frac{2}{5}\cdot\delta_0 \in [\delta_{min}, \delta_{max}]$.)

6. $f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$

a) Määritä $\frac{\partial f}{\partial x}(x,y)$ ja $\frac{\partial f}{\partial y}(x,y)$, kun $(x,y) \neq (0,0)$. (2p.)

b) Määritä $\frac{\partial f}{\partial x}(0,0)$ ja $\frac{\partial f}{\partial y}(0,0)$ käyttämällä osittaisderivaatan määritelmää. (2p.)

c) Määritä $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ ja $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ origossa käyttämällä osittaisderivaatan määritelmää. (2p.)

Ratkaisu: a) $f_x = \frac{\partial f}{\partial x} = \frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)2x}{(x^2+y^2)^2} = \frac{4x^2y^3-y^5+x^4y}{(x^2+y^2)^2}$,
$f_y = \frac{\partial f}{\partial y} = \frac{(x^3-3xy^2)(x^2+y^2)-(x^3y-xy^3)2y}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$, kun $(x,y) \neq (0,0)$.

b) $f_x(0,0) = \frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}(f(0+\Delta x, 0) - f(0,0)) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}(0-0) = 0$,
$f_y(0,0) = \frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}(f(0, 0+\Delta y) - f(0,0)) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}(0-0) = 0$.

c) $f_{xy}(0,0) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(0,0) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}(f_x(0,0+\Delta y) - f_x(0,0)) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}(\frac{-(\Delta y)^5}{(\Delta y)^4} - 0) = -1$,
$f_{yx}(0,0) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(0,0) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}(f_y(0+\Delta x, 0) - f_y(0,0)) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}(\frac{(\Delta x)^5}{(\Delta x)^4} - 0) = 1$.
(Huom: $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0)$.)