MS-E1652 Computational methods for differential equations

Course exam and Exam, 9:00–13:00, December 7, 2020 **Hyvönen/Hirvi**

Access to extra materials and resources is not restricted, but consulting other people is strictly forbidden. All solutions must indicate how the end results and conclusions were reached via calculations by hand, with a credible number of intermediate stages presented.

The grade based on the combination of the course exam and the exercise points is deduced by accounting for the four solutions that produce the most points; if a student, e.g., gets 6, 2, 4, 4 *and* 5 *points from the individual exam problems, then the number of exam points considered in connection to the exercise points is* $19 = 6 + 4 + 4 + 5$. *The grade based on the mere final exam is deduced by accounting for all five solutions. The better of the two alternatives will correspond to the final grade.*

1. Consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = 1, \tag{1}$$

for $t \geq 0$.

(a) Let $f(t, x) = x \sin(t)$ and prove that this function satisfies the global Lipschitz condition with respect to its second variable. Give the unique solution of (1) explicitly.

(b) Let $f(t, x) = x^2 \sin(t)$ and once again solve (1) explicitly. Is the solution well defined for all $t \in [0, \infty)$? Does this f satisfy the (global) Lipschitz condition with respect to its second variable?

2. Consider the *linear multistep method* (LMM)

$$x_{j+2} - \frac{1}{2}x_{j+1} - \frac{1}{2}x_j = \frac{3}{2}hf_{j+2}, \qquad j = 0, 1, 2, \dots\ . \tag{2}$$

Consider the following questions/tasks. Justify your answers.

(a) Is (2) an explicit or an implicit LMM?

(b) Prove that (2) is consistent of order $p = 1$.

(c) Prove that (2) is not consistent of order $p = 2$.

(d) Is (2) zero-stable?

(e) Why is zero-stability an indispensable property of any functional LMM?

(f) Does the point $\hat{h} = -10$ belong to the region of absolute stability for (2)?

3. Consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0 \neq 0. \tag{3}$$

and the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{1}{4}h(k_1 + 3k_2), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + \frac{2}{3}h, x_j + \frac{2}{3}hk_1\Big). \end{aligned} \tag{4}$$

(a) Is (4) an explicit or an implicit RK method? Why?

(b) Let $f(t, x) = \lambda x$ with $\lambda \in \mathbb{C}$. Write x_j produced by (4) with the help of x_0 for an arbitrary $j \in \mathbb{N}$.

(c) Let $f(t, x) = \lambda x$ with $\lambda \in \mathbb{C}$. Prove that

$$x_1 = x(h) + O(h^3)$$

where $x : \mathbb{R}_+ \to \mathbb{C}$ is the exact solution of (3) for the considered f. Based on this result, what seems to be the order of the RK method (4)?

(d) Let $f(t, x) = \lambda x$ with a fixed *real* $\mathbb{R} \ni \lambda < 0$. For which values of the time step $h > 0$ does the numerical solution produced by (4) satisfy $x_j \to 0$ as $j \to \infty$?

4. Consider the initial value problem

$$u'(t) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} u(t), \qquad u(0) = u_0. \tag{5}$$

(a) Prove that

$$\lim_{t\to\infty} u(t) = 0 \in \mathbb{R}^2$$

if and only if $u_0 = [c, 0]^{\mathrm{T}}$ for some $c \in \mathbb{R}$.

(b) Let $u_0 = [1, 0]^{\mathrm{T}}$ and apply the explicit Euler's method to (5). For what values of the time step $h > 0$ does the produced numerical solution satisfy

$$\lim_{j\to\infty} u_j = 0 \in \mathbb{R}^2 \ ?$$

5. When the initial/boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,1), \end{cases}$$

is spatially discretized by the standard central second order difference approximation, one ends up with the following initial value problem:

$$U'(t) = A\,U(t), \qquad U(0) = G, \tag{6}$$

for all $t \geq 0$. Here, $G = [g(x_1), \ldots, g(x_m)]^{\mathrm{T}}$ and $U(t) \approx [u(x_1,t), \ldots,\ u(x_m,t)]^{\mathrm{T}}$, with $x_j = jh$ and $h = 1/(m+1)$ being the mesh parameter.

(a) Write down the matrix $A \in \mathbb{R}^{m\times m}$. (It is enough describe the structure of A — you need not present an actual proof.)

(b) Introduce Heun's method for numerically solving (6). Let $\delta > 0$ be the time step size and denote by U_k the approximation of $U(k\delta)$ for $k = 0, 1, 2, \ldots$.

(c) For which $\delta > 0$ is the introduced method (for sure) absolutely stable, that is,

$$\lim_{k\to\infty} U_k = 0 \in \mathbb{R}^m$$

for any $G \in \mathbb{R}^m$? Justify your answer.

(d) Let $T > 0$ be fixed. What is the largest value of $p \in \mathbb{N}$ for which the estimate

$$\left|U_k - U(k\delta)\right| \leq C\delta^p \qquad \text{for all } G \in \mathbb{R}^m$$

holds for any $k \in \mathbb{N}$ such that $k\delta \in (0, T]$ and some $C > 0$ that may depend on m. Is it possible to choose C to be independent of m? (You need not deduce the order of Heun's method, but you can refer, e.g., to the lecture notes for that information.)