Aalto University School of Science
Department of Mathematics and Systems Analysis — Korte

MS-C1350 Partial differential equations, fall 2020

Course exam on 14 Dec 2020 at 9:00-13:00

This set of problems is for the participants of the course in the fall 2020 and affects 40% in the grading of the course.

Remember to explain carefully your answers. Remember to answer to Problem 1 (multiple choice question) as well.

Each question is worth 6 points. Some problems may have bonus parts with extra points.

2. (a) Assume that $u = u(x,t)$ is a solution to $u_t - c^2\Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$, with $c > 0$. For which values of parameters $a, b \in \mathbb{R}$ the function $v(x,t) = u(ax, bt)$ is a solution to $v_t - \Delta v = 0$? (4p.)

   (b) How does the situation change, if u is a solution to the initial value problem
$$\begin{cases} u_t - c^2\Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$
and we want to find parameters $a, b \in \mathbb{R}$ such that $v(x,t) = u(ax, bt)$ satisfies the initial value problem
$$\begin{cases} v_t - \Delta v = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = f(x), & x \in \mathbb{R}^n? \end{cases} \qquad \text{(2p.)}$$

3. (a) (3p.) Suppose that $\Omega \subset \mathbb{R}^n$ is a connected and bounded domain and $f, g \in C^\infty(\mathbb{R}^n)$. Give all solutions to the problem
$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on} \quad \partial\Omega, \end{cases}$$
if we know that functions u_1 and u_2 are solutions to the following problems
$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega, \\ \frac{\partial u_1}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$
and
$$\begin{cases} -\Delta u_2 = 0 & \text{in} \quad \Omega, \\ \frac{\partial u_1}{\partial \nu} = g & \text{on} \quad \partial\Omega. \end{cases}$$

   (b) (3p.) Assume that ϕ and ψ are smooth enough functions. Find a solution to the following problem by applying a suitable reflection method and d'Alembert's formula:
$$\begin{cases} v_{tt} - v_{xx} = 0, & 0 < x < \infty,\ 0 < t < \infty, \\ v(x,0) = \phi(x), v_t(x,0) = \psi(x), & 0 < x < \infty, \\ v_x(0,t) = 0, & 0 < t < \infty. \end{cases}$$

4. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and $f \in C^2(\mathbb{R}^2)$. Let us consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ in Ω with boundary values $u(x,0) = 0$, $u(x,1) = f(x)$, when $0 < x < 1$, and $u(0,y) = u(1,y) = 0$, when $0 < y < 1$.

   (a) Reduce the problem to two ODEs by using separation of variables. (2p.)

   (b) Solve the separated equations to find special solutions. (2p.)

   (c) Let $f = \sin(3\pi x)$. Find the explicit solution u. (2p.)

   (BONUS) How would you find a solution with correct boundary values with the general Dirichlet boundary condition $u = f$ on $\partial\Omega$? (2p.)

5. Let u be a solution to the problem $u_{tt} - c^2 \Delta u = f$ in $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded domain, $c > 0$ and f is a smooth enough function.

   (a) Show that the energy

$$e(t) = \frac{1}{2} \int_\Omega \big((u_t)^2 + c^2 |\nabla u|^2 \, dx\big)$$

   is preserved by u. (3p.)

   (b) Use the result from part (a) to prove that there exists at most one solution to the problem

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in} \quad \Omega_T, \\ u = g & \text{on} \quad \Gamma_T, \\ u_t = h & \text{in} \quad \Omega \times \{t = 0\}. \end{cases} \qquad \text{(3p.)}$$

   (It is ok to assume that solutions are smooth enough so that all necessary derivatives exist and integrals are finite.)