MS-E1461 Hilbert spaces (Aalto University, Turunen)
Examination Tuesday 15.12.2020, 9:00-13:00

Points also for good effort! You may use websites and the course material. Yet, the problems must be solved individually. Please indicate the sources you used.

1. Let G be a Hilbert space over the real scalar field $\mathbb{R}$. Explain how the Cartesian product $H = G \times G = \{(u,v) : \ u,v \in G\}$ can be equipped with a complex Hilbert space structure: What are the vector space operations? What is the inner product? What is the norm? Verify your claims!

   (Hint: Think of $(u,v) \in H$ as $u + \mathrm{i}v$.)

2. Consider the Hilbert space $H = \ell^2(\mathbb{Z}^+)$ of absolutely square summable functions $u : \mathbb{Z}^+ \to \mathbb{C}$. For $u \in H$ and $N \in \mathbb{Z}^+$, define $P_N u : \mathbb{Z}^+ \to \mathbb{C}$ by

$$P_N u(x) := \begin{cases} u(x) & \text{if} \quad x \le N, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Let $P := P_N$. Show that $P \in \mathscr{B}(H)$ such that $P^2 = P = P^*$ (meaning that P is an orthogonal projection).

   (b) Show that $\|P_N u - u\| \to 0$ as $N \to \infty$, for all $u \in H$.
   Show that $\|P_N - I\| = 1$ for all $N \in \mathbb{Z}^+$.

   (c) Apply the Gram-Schmidt orthonormalization process for a linearly independent sequence $(u_k)_{k=1}^\infty$ in H, where $u_k(x) := \begin{cases} (-1)^k/x & \text{if } x \le k, \\ 0 & \text{if } x > k. \end{cases}$

3. (a) Show that for each $A \in \mathscr{B}(H)$ there exists $A^* \in \mathscr{B}(H)$ such that $\langle Au, w \rangle = \langle u, A^* w \rangle$ for all $u, w \in H$.

   (b) Let $H = \ell^2(\mathbb{Z})$ be the Hilbert space of the absolutely square summable functions $u : \mathbb{Z} \to \mathbb{C}$. How do the bounded linear functionals $\varphi : H \to \mathbb{C}$ look like?

   (c) Let $L^2(\mathbb{R})$ be the Hilbert space of the absolutely square integrable functions $u : \mathbb{R} \to \mathbb{C}$. Let H be the Hilbert space of the Hilbert-Schmidt operators $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. How do the bounded linear functionals $\psi : H \to \mathbb{C}$ look like?

4. Think of a direct sum decomposition $H = \bigoplus_{\alpha \in J} H_\alpha$, and let $P_\alpha : H \to H$ be the orthogonal projection onto H_α. Suppose $|\varphi(\alpha)| < 1$ for all $\alpha \in J$, where $\varphi : J \to \mathbb{C}$ is injective. For $u \in H$, let $Au := \sum_{\alpha \in J} \varphi(\alpha)\, P_\alpha u$.

   (a) Show that $A \in \mathscr{B}(H)$. When would A be self-adjoint?
   (b) What is the spectrum $\sigma(A)$ of A here?
   Which are the eigenvalues and the corresponding eigenvectors?
   Can $\sigma(A)$ here contain something else than the eigenvalues?
   (Justify your answers! You may use the fact that $\sigma(A)$ is closed.)