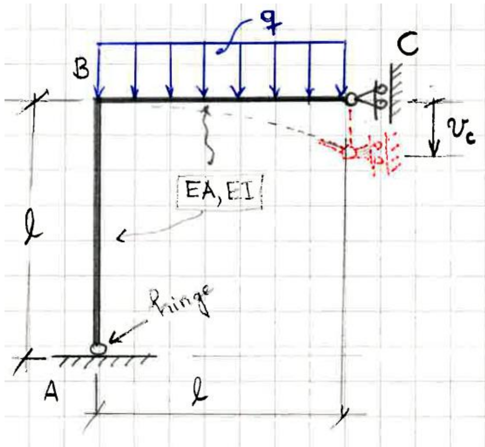
CIV-E1020 - Mechanics of Beam and Frame Structures - <u>duration: 3h</u>

It is compulsory to solve only THREE (3) EXERCISES that you choose freely: only three exercises will be graded

Formulary enclosed with the questions

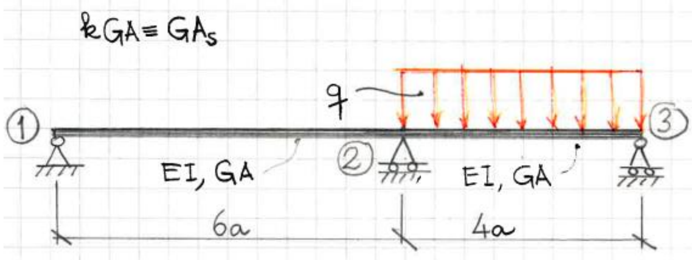
Examination 20.10.2020

The material is linear elastic in all the exam structures in this examination.

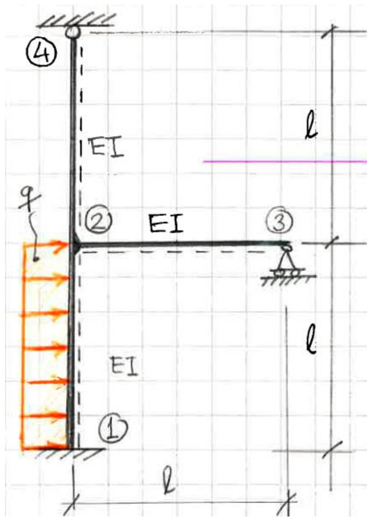


1. Use the dummy unit-load theorem (or method) and determine the vertical displacement at the roller *C*. Account both for effects of bending and axial forces. [5p]

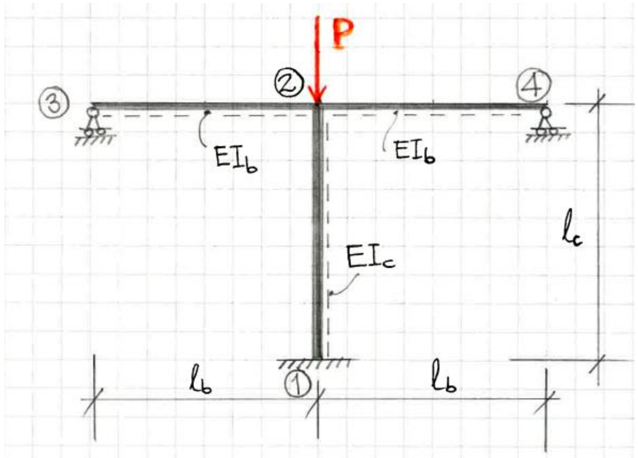
Grading 3 obligatory exercises)		GRADE
Points in this exam		
14.25	15	5
12.75	13.5	4
9.75	12	3
8.25	9	2
6	7.5	1
< 6	fail	0



2. Use the general force method and determine bending moment at the mid-support 2 for the continuous beam. Account for both effects of bending and shearing [5 p] (if someone does not account for shearing and accounts only for bending → he will get 4 points]



3. Use Slope-Deflection Method and determine the bending moment at the clamping support *1.* [4 points]
Determine and draw the bending moment diagram [1 point]



4. Use Slope-Deflection Method and derive the expression of the *criticality condition* to determine the critical buckling load for this continuous column [5p]. The bending rigidity of the column is twice the bending rigidity of the beam; $EI_c = 2 * EI_b$.

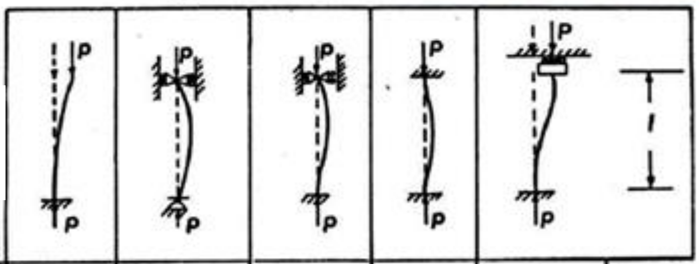
No need to solve numerically for the buckling load. The force *P* is centrically applied to the column.

The slope-deflection method:

Euler's basic buckling cases

Eulerin perusnurjahdus

$$P_{cr} = \mu \frac{\pi^2 EI}{\ell^2}$$



Tapaus	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$M_{ij} = A_{ij}\phi_{ij} + B_{ij}\phi_{ji} - C_{ij}\psi_{ij} + \bar{M}_{ij} \qquad M_{ij}^0 = A_{ij}^0 \varphi_{ij} - C_{ij}^0 \psi_{ij} + \overline{M}_{ij}^0$$

Beam-column with constant flexural rigidity:

$$A_{ij} = A_{ji} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)} \frac{6EI}{L}, \qquad B_{ij} = B_{ji} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)} \frac{6EI}{L}$$

$$C_{ij} = A_{ij} + B_{ij} \qquad A_{ij}^0 = C_{ij}^0 = \frac{1}{\psi(kL)} \frac{3EI}{L},$$

$$kL \equiv L\sqrt{\frac{P}{EI}}$$

Berry's functions:

Olkoon $\lambda \equiv kL$,

$$\lambda \equiv kL$$

Puristettu sauva:

Compression:

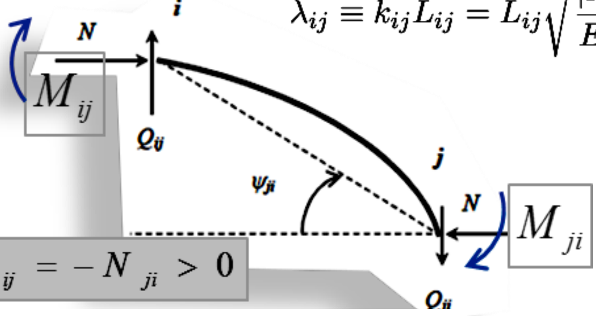
$$\phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(\tan\frac{\lambda}{2} - \frac{\lambda}{2}\right),$$

Vedetty sauva:

Extension:

$$\phi(\lambda) = \frac{6}{\lambda}\left(-\frac{1}{\sinh\lambda} + \frac{1}{\lambda}\right), \quad \psi(\lambda) = \frac{3}{\lambda}\left(-\frac{1}{\lambda} + \frac{1}{\tanh\lambda}\right), \quad \text{ja} \quad \chi(\lambda) = \frac{24}{\lambda^3}\left(-\tanh\frac{\lambda}{2} + \frac{\lambda}{2}\right),$$

$$\lambda_{ij} \equiv k_{ij}L_{ij} = L_{ij}\sqrt{\frac{|N_{ij}|}{EI_{ij}}} \rightarrow \lambda \equiv kL = L\sqrt{\frac{|N|}{EI}}$$

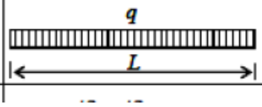


$$N \equiv -N_{ij} = -N_{ji} > 0$$

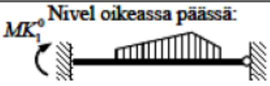
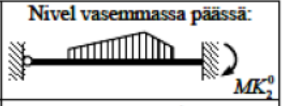
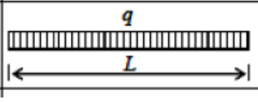
$$Q_{ij} = Q_{ij}^0 - (M_{ij} + M_{ji})/L - N\psi_{ij}$$

$$\overline{M}_{12} \equiv MK_1$$

$$\overline{M}_{ij}, \overline{M}_{ji}$$

N:o	Kuormitus	Kiinnitysmomentit: 
1		$MK_1 = -\frac{qL^2}{12}$, $MK_2 = \frac{qL^2}{12}$
2		

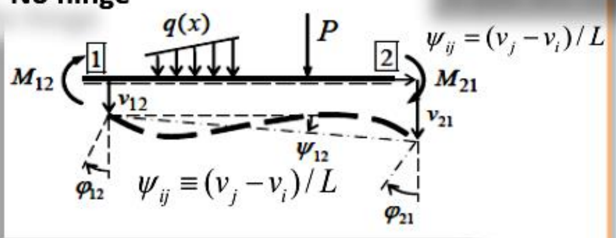
$$\overline{M}_{ij} \qquad \overline{M}_{ji}$$

N:o	Kuormitus	Nivel oikeassa päässä: 	Nivel vasemmassa päässä: 
1		$MK_1^0 = -\frac{qL^2}{8}$	$MK_2^0 = \frac{qL^2}{8}$

The stiffness equation relating the end-moments to the end-displacements

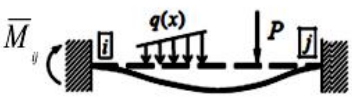
If you are using lecture's notations

No hinge



$$M_{ij} = a_{ij}\varphi_{ij} + b_{ij}\varphi_{ji} - c_{ij}\psi_{ij} + \overline{M}_{ij}, \quad i \neq j$$

$$a_{ij} = \frac{4EI}{L}, \quad b_{ij} = \frac{2EI}{L}, \quad c_{ij} = \frac{6EI}{L} \qquad (EI\text{-constant})$$



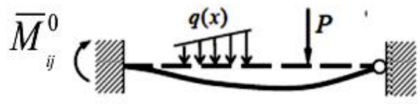
Fixed end-moment resulting from external mechanical loading, look from tables

One node is hinged

The is a superscript "0" means that the support at end *j* is hinged

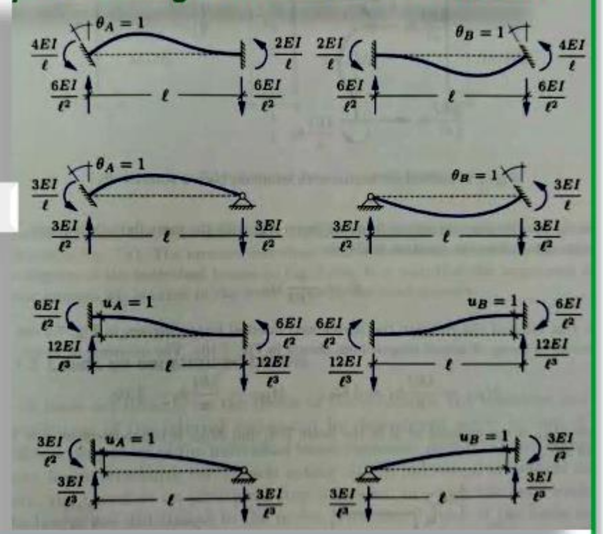
$$M_{ij}^0 = a_{ij}^0\varphi_{ij} - c_{ij}^0\psi_{ij} + \overline{M}_{ij}^0$$

$$a_{12}^0 = c_{12}^0 = \frac{3EI}{L} \qquad \psi_{ij} = (v_j - v_i)/L$$



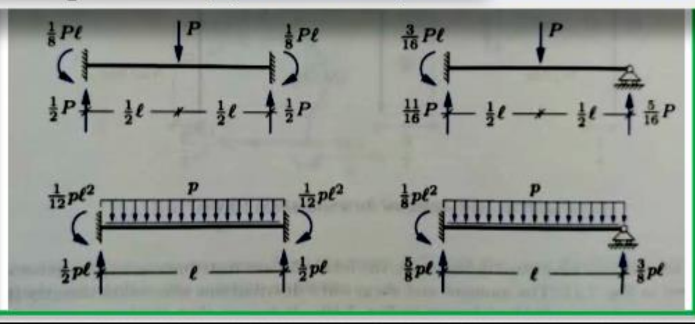
Fixed end-moment resulting from external mechanical loading, look from tables

If you are using Krenk's textbook notations



If you are using Krenk's textbook notations

Bending Moments (pay attention to the sign convention to convert to Fixed-End-Moments)



Maxwell-Mohr integrals table

TABLEAU DES INTEGRALES $\int_0^{\ell} {}^{i}M\,{}^{k}M\,dx$

${}^{k}M$ \ ${}^{i}M$	c, ℓ	c, ℓ	d, ℓ	c, d, ℓ	2ème deg. c, ℓ
a, ℓ	$ac\ell$	$\frac{1}{2}ac\ell$	$\frac{1}{2}ad\ell$	$\frac{1}{2}a\ell(c+d)$	$\frac{2}{3}ac\ell$
a, ℓ	$\frac{1}{2}ac\ell$	$\frac{1}{3}ac\ell$	$\frac{1}{6}ad\ell$	$\frac{1}{6}a\ell(2c+d)$	$\frac{1}{3}ac\ell$
b, ℓ	$\frac{1}{2}bc\ell$	$\frac{1}{6}bc\ell$	$\frac{1}{3}bd\ell$	$\frac{1}{6}b\ell(c+2d)$	$\frac{1}{3}bc\ell$
a, b, ℓ	$\frac{1}{2}(a+b)c\ell$	$\frac{1}{6}(2a+b)c\ell$	$\frac{1}{6}(a+2b)d\ell$	$\frac{\ell}{6}[a(2c+d)+b(c+2d)]$	$\frac{1}{3}(a+b)c\ell$
2ème deg. a, ℓ	$\frac{1}{3}ac\ell$	$\frac{1}{4}ac\ell$	$\frac{1}{12}ad\ell$	$\frac{1}{12}a\ell(3c+d)$	$\frac{1}{5}ac\ell$
2ème deg. a, ℓ	$\frac{2}{3}ac\ell$	$\frac{5}{12}ac\ell$	$\frac{1}{4}ad\ell$	$\frac{1}{12}a\ell(5c+3d)$	$\frac{7}{15}ac\ell$

Most common student's mistakes / errors in exam of 20/10/2020

↖ do not do them again... improve

- #1 **Physical units not consistent:**

like (example from exam) →

$$5\,q\cdot a + q\,a^2 + 28 + 800\,a^3 q^3$$

$$\frac{N}{m}\cdot m + \frac{N}{m}\cdot m^2 + [\cdot] + m^3\left[\frac{N}{m}\right]^3 = \,!$$

$$1\text{ cat} + 2\text{ dogs}^2 + 3 + 6\text{ cats}^3\cdot\text{dogs}^3 = ?$$

⇒ EXAM MARK = 0 pt or max. 1 if any logic

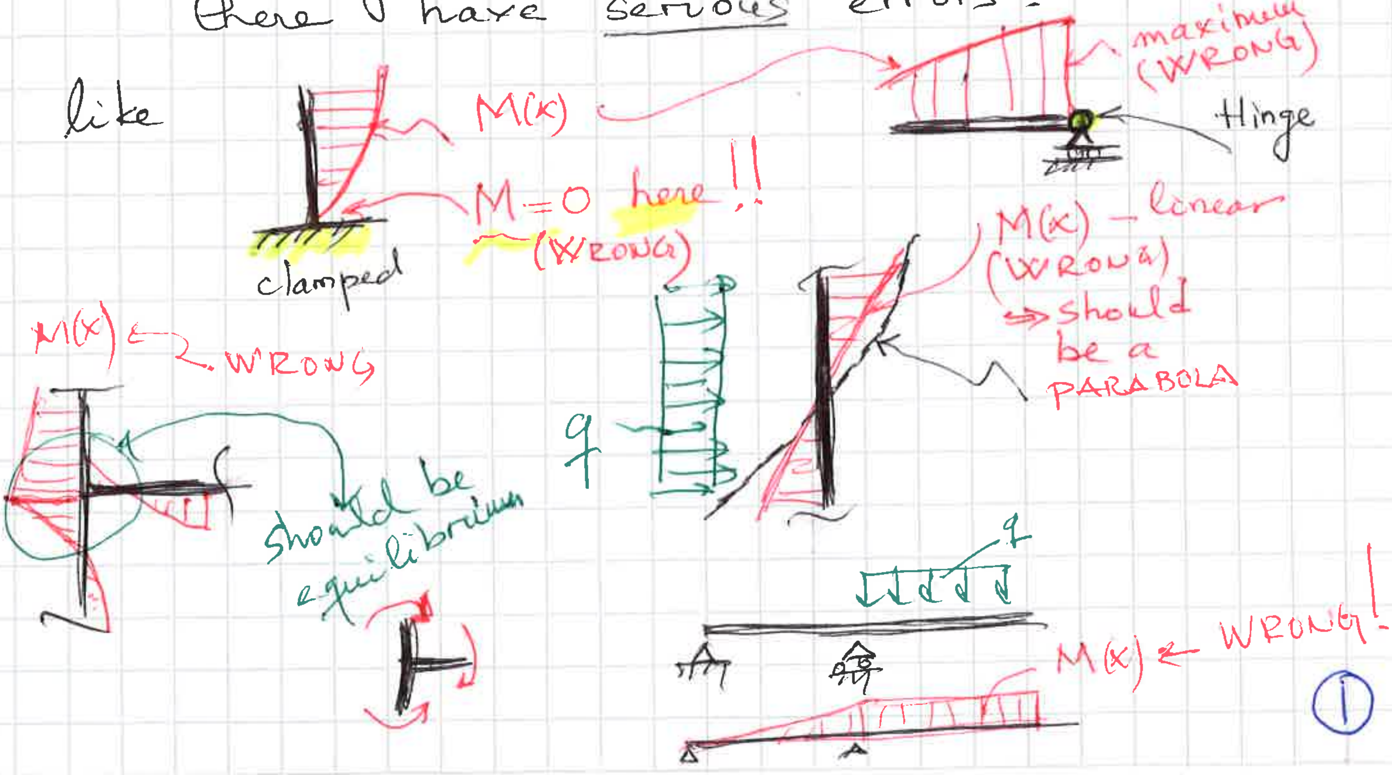
∴ Always check that the equations you are writing (sometimes or you're reading) are dimensionally consistent

- #2 **The statics is WRONG** → In statically determined structure !
  - Reactions & member forces are still not correctly determined !

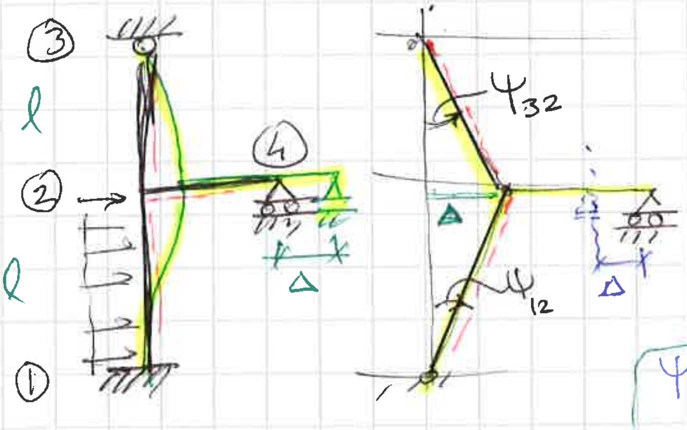
∴ Train again & again : FBD + Equations of equilibrium

  - Bending moment diagrams still here and there have serious errors.

like



#3 (SWAY AND NON-SWAY Frame — the difference between, not yet understood



$\psi_{12} = \psi_{21} = -\psi_{32} = -\psi_{23} \equiv \psi$

∴ one time sway

• Note that $\psi_{12} = \psi_{21} > 0$

and $\psi_{32} = \psi_{23} < 0$

$\psi = \Delta/\ell$ ② ① $\varphi_{ji}, \psi_{ji} > 0$

$\psi_{ij}, \varphi_{ij} > 0$

#3.1 — If one assumes a non-sway for a sway frame then this is A serious error (ajatusvirhe)

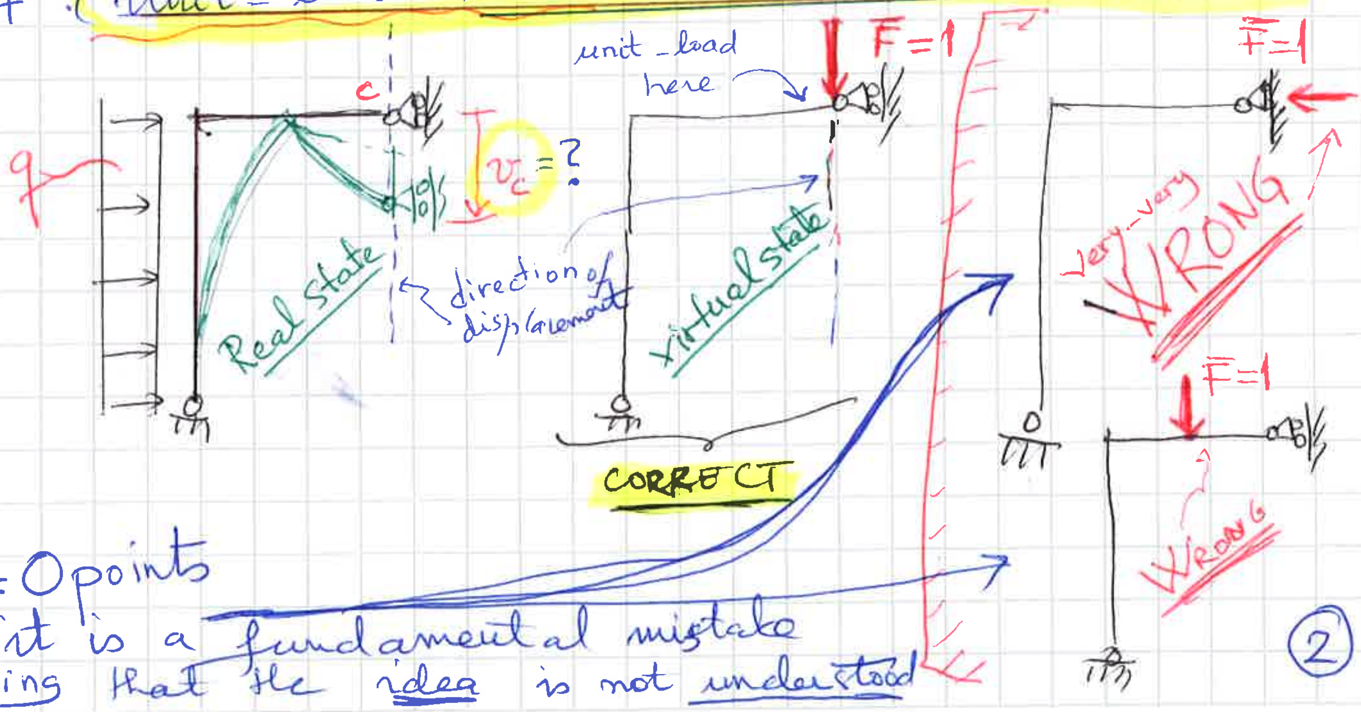
⇒ in exam, the solution which is naturally wrong will be graded at best by 1 pnt/5

(It is simply WRONG → Mark = ≤ 1 point)

IN EXAM MARK = 0 POINTS OR MAX. 1pnt

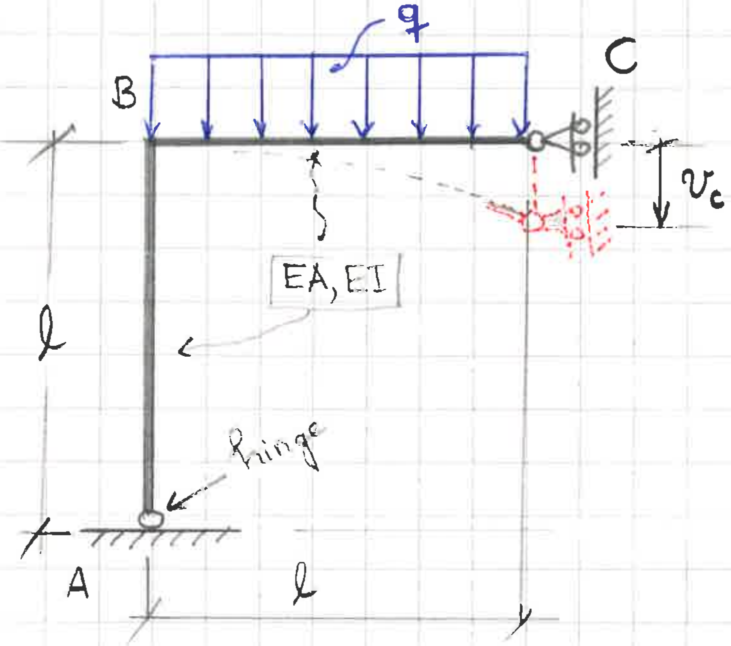
#3.2 — be careful with the sign of ψ_{ij} terms (their sign is like the sign of rotation φ_{ij})

• #4 (virtual) Unit-load theorem ... to determine displacement

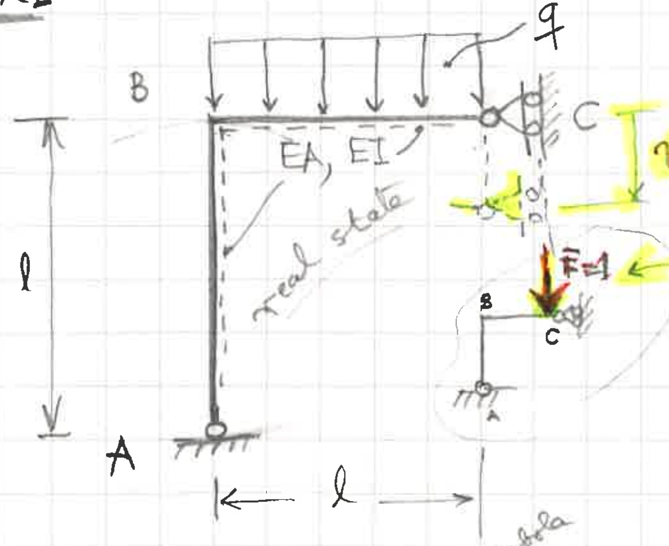


Mark = 0 points since it is a fundamental mistake showing that the idea is not understood

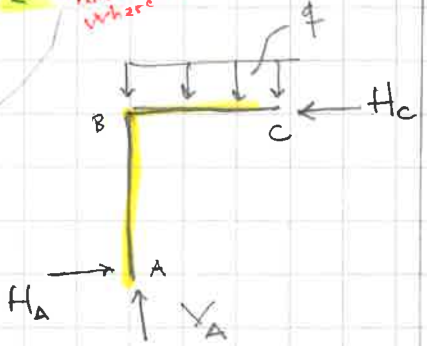
Ex 1



EX1:



$$1 \cdot v(C) = \sum \int_l \bar{M} \underset{M/EI}{\kappa}\, ds + \sum \int_l \bar{N} \underset{N/EA}{\varepsilon^\circ}\, ds$$



$A \circlearrowright: \; q\frac{l^2}{2} - H_C \cdot l = 0$

$H_C = ql/2$

$\rightarrow: \; H_A = H_C = ql/2$

$C \circlearrowright:$

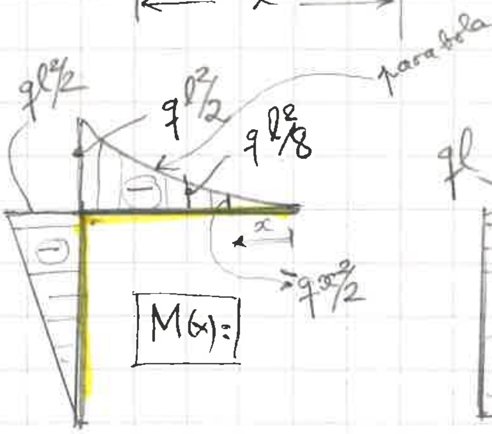
$H_A \cdot l - V_A \cdot l + q\frac{l^2}{2} = 0$

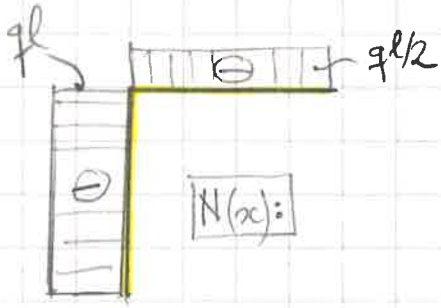
$V_A = H_A + ql/2$

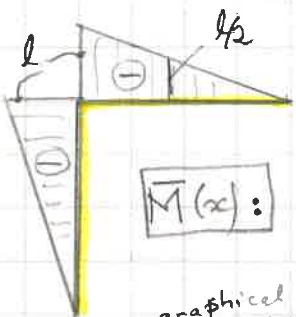
$= ql/2 + ql/2 = ql$

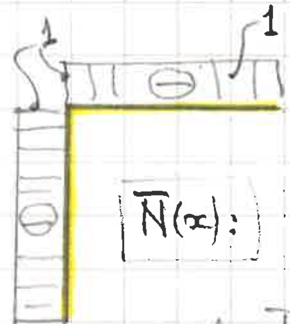
$\uparrow: \; V_A - ql = 0$ ok

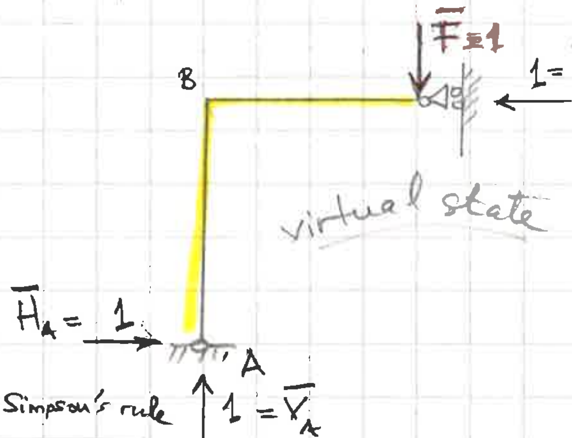
(cross-check)











graphical integration — numerical integration: Simpson's rule

$$v(C) = \frac{1}{EI}\Big[\underbrace{\tfrac{1}{2}l^2}_{\bar{M}} \cdot \tfrac{2}{3} \cdot \underbrace{q\tfrac{l^2}{2}}_{M}\Big] + \frac{l}{6EI}\Big[q\frac{l^2}{2} \cdot l + 4 \cdot q\frac{l^2}{8} \cdot \frac{l}{2} + 0 \cdot 0\Big] + \frac{1}{EA}\Big[ql^2 + q\frac{l^2}{2}\Big] =$$

$$= \frac{1}{EI} q l^4 \cdot \frac{1}{6} + \frac{ql^4}{EI} \cdot \Big[\underbrace{\frac{1}{2} + \frac{1}{4}}_{=3/4}\Big] \cdot \frac{1}{6} + \frac{ql^2}{EA} \cdot \Big[\underbrace{1 + \frac{1}{2}}_{=3/2}\Big]$$

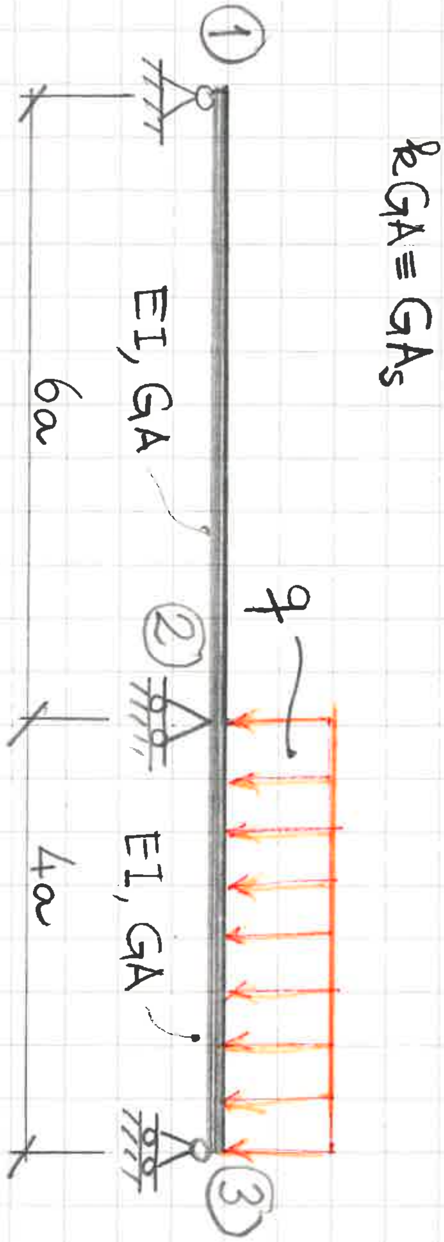
$$= \frac{ql^4}{EI} \cdot \frac{7}{24} + \frac{ql^2}{EA} \cdot \frac{3}{2} = \frac{7}{24}\frac{ql^4}{EI} \cdot \Big[1 + \frac{EI}{EAl^2} \cdot \frac{3}{2} \cdot \frac{24}{7}\Big]$$

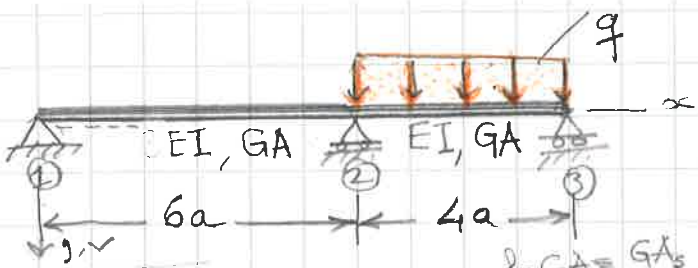
$$= \boxed{\frac{7}{24}\frac{ql^4}{EI}} \cdot \Big[1 + \underbrace{\frac{36}{7}\frac{EI}{EAl^2}}\Big] \approx 0{,}292\, ql^4/EI \cdot \Big(1 + 5{,}14 \frac{EI}{EAl^2}\Big)$$

bending only part

effect of axial force (stretching)

Ex 2





1 — determine the support M bending moment *) [3pnt or 4pnt] M ↗ Q

2 — draw the bending moment distribution [1 pnts] (force method)

*) a) not accounting for the effect of sharing [3Pnts]

* b) accounting for the effect of sharing [4 pnts]

$m_s = 1$

$$\delta_{11} X_1 + \delta_{10} = 0$$

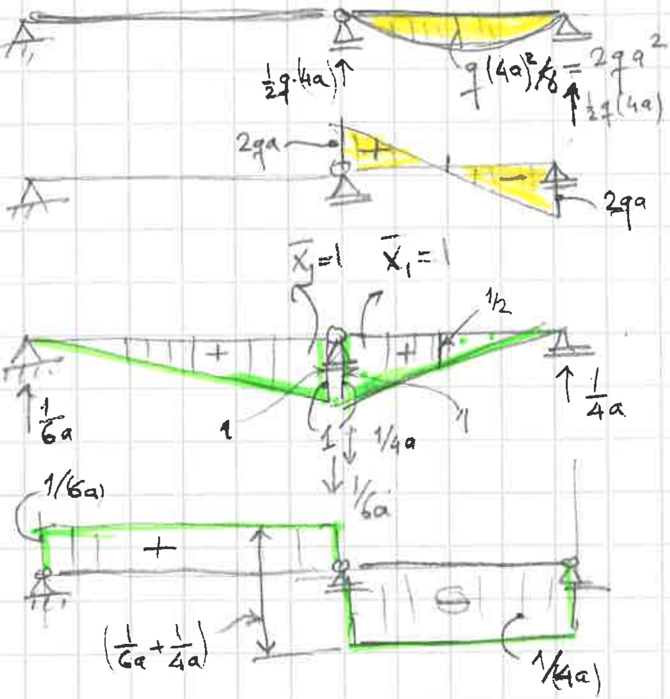
$$\delta_{11} = \Sigma \int_l \bar{M}_1^2/EI \, ds + \Sigma \int_l \frac{\bar{Q}_1^2}{kGA} ds$$

$$\delta_{10} = \Sigma \int_l \bar{M}_1 M_0/EI \, ds + \Sigma \int_l \frac{\bar{Q}_1 Q_0}{kGA} ds$$

$$M(x) = M_0 + X_1 \cdot \bar{M}_1(x)$$

$$q(4a)^2/8 = \frac{16}{8} \cdot q a^2 = 2qa^2$$

$$\frac{1}{2} q \cdot (4a) = 2qa$$



$$\delta_{11} = \frac{1}{EI}\left[\underbrace{\frac{1}{2} \cdot 6a \cdot \frac{2}{3}}_{=2a} + \underbrace{\frac{1}{2} \cdot 4a \cdot \frac{2}{3}}_{=4/3a}\right] + \frac{1}{kGA}\left[\frac{1}{(6a)^2} \cdot 6a + \frac{1}{(4a)^2} \cdot 4a\right] =$$

$$= \frac{1 \cdot a}{EI}\left(2 + \frac{4}{3}\right) + \frac{1}{kGA} \cdot \left(\frac{1}{6a} + \frac{1}{4a}\right)$$

$$= \frac{a}{EI} \cdot \frac{10}{3} + \frac{1}{kGAa} \cdot \frac{5}{12}$$

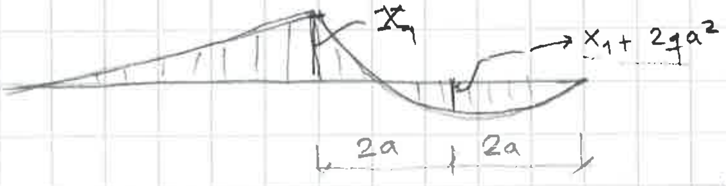
$$\delta_{10} = \frac{1}{EI} \cdot \frac{4a}{6} \cdot 4 \cdot (2qa^2) \cdot \frac{1}{\cancel{2}} + \frac{1}{kGA} \cdot \underbrace{[\ \text{antisym.} * \text{sym.}\ ]}_{=0}$$

$$= \frac{qa^3}{EI} \cdot \frac{8}{3} = \frac{8}{3} \frac{qa^3}{EI}$$

only bending — with shear

$$X_1 = -\delta_{10}/\delta_{11} = -\frac{8}{3}\frac{qa^3}{EI} \cdot \frac{1}{\frac{10}{3}\frac{a}{EI} + \frac{5}{12}\frac{1}{kGAa}} = \boxed{-\frac{8}{3} q a^2} \cdot \frac{1}{\frac{10}{3} + \frac{5}{12}\frac{1}{kGAa^2}}$$

$$X_1 = (\text{without sharing}) = -\frac{8}{3}\frac{qa^{\cancel{3}\,2}}{\cancel{EI}} \cdot \frac{3\cancel{EI}}{10\cancel{a}} = -\frac{\cancel{3}}{10} \cdot \frac{8}{\cancel{3}} q a^2 = -8qa^2/10$$



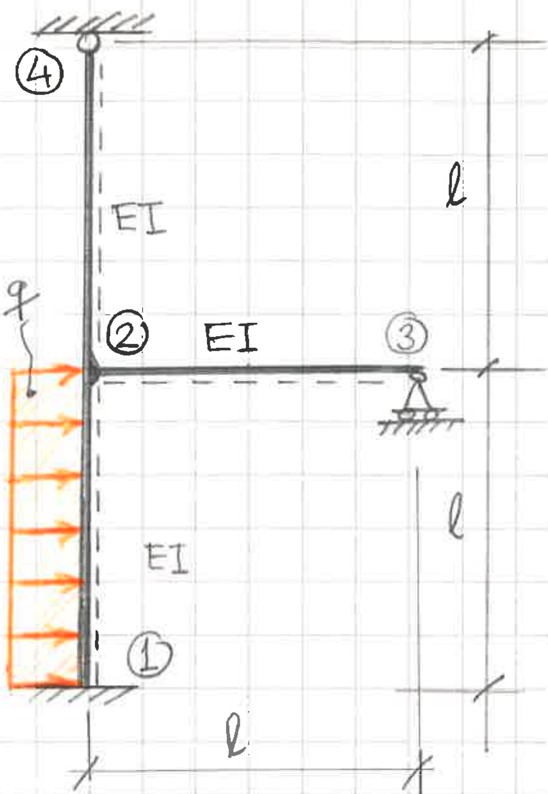
M:

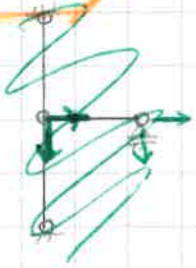
* 1pnts: think what are the possible primary released structure like?
2) choose the less [illegible] MDM-method

Slope deflection method

EX3

- determine bending moment M_1 at the clamping node 1. [4pnts]
- determine and draw the bending moment distribution. [1p]





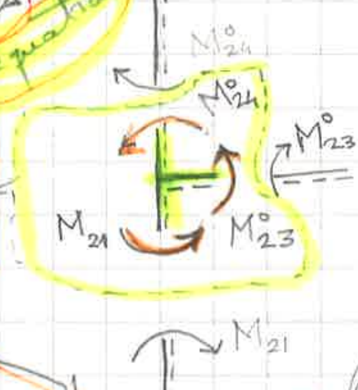
$n_k = 2$, ONCE-SWAY

$$\begin{cases} \varphi_{12} = 0, \quad \psi_{23} = \psi_{32} = 0 \\ \varphi_{21} = \varphi_{23} = \varphi_{24} \equiv \varphi_2 \\ \psi_{21} = \psi_{12} = -\psi_{24} = -\psi_{42} \equiv \psi \end{cases}$$

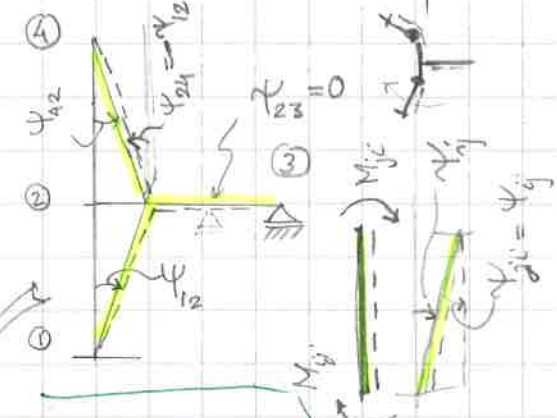
∴ two kin. unknowns (independent) : φ_2 and ψ

needs

two equilibrium equations (1) and (2)



$$(1) \quad \boxed{M_{21} + M^0_{23} + M^0_{24} = 0}$$



Summa Summarum:

$$(1): \quad M_{21} + M^0_{23} + M^0_{24} = 0$$
$$(2): \quad M_{21} + M_{12} - M^0_{24} + ql^2/2 = 0$$

$$\begin{bmatrix} a_{21} + a^0_{23} + a^0_{24} & -(c_{21} + c^0_{24}) \\ a_{21} + b_{12} - a^0_{24} & -(c_{21} + c_{12} + c^0_{24}) \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \psi \end{bmatrix} = -\frac{ql^2}{12} \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

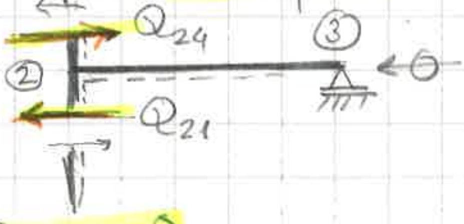
$$\begin{bmatrix} 10 & -9 \\ \cancel{-9}\ -3 & 15 \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \psi \end{bmatrix} = -\frac{ql^3}{12} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a) $\delta W_{int} + \delta W_{ext} = 0, \ \forall \delta\psi$ $\forall \delta\psi$

$$(M_{21}\delta\psi + M_{12}\delta\psi + M^0_{24} \cdot (-\delta\psi)) + ql \cdot \frac{l}{2} \delta\psi = 0,$$

$$\Rightarrow \boxed{M_{21} + M_{12} - M^0_{24} + ql^2/2 = 0} \quad (2)$$

Equivalently shear equations



compare (of course the same!)

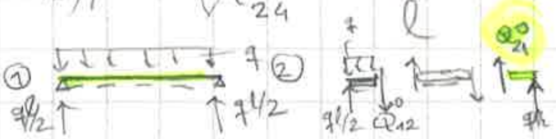
$$\boxed{-q\frac{l^2}{2} - M_{12} - M_{21} + M^0_{24} = 0} \quad (2)$$

or

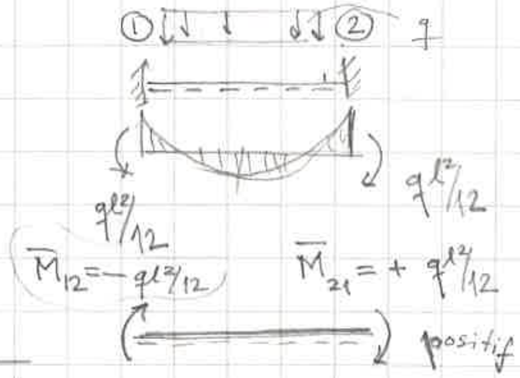
b) $Q_{21} - Q_{24} = 0$

$$0 = Q^0_{21} - (M_{21} + M_{12})/l - \left(Q^0_{24} - \frac{M^0_{24} + M^0_{42}}{l}\right)$$

$-ql/2$ $\quad Q^0_{21}$



1



$$M_{21} = a_{21}\varphi_2 - c_{21}\psi + \underbrace{ql^2/12}_{\overline{M}_{21}}$$

$+$

$$M^\circ_{23} = a^\circ_{23}\varphi_2$$

$+$

$$M^\circ_{24} = a^\circ_{24}\varphi_2 - c^\circ_{24}(\overbrace{-\psi}^{\psi_{24}=-\psi_{12}=\psi})$$

$$(1') \Rightarrow (\underset{4\frac{EI}{l}}{a_{21}} + \underset{\frac{3EI}{l}}{a^\circ_{23}} + \underset{\frac{3EI}{l}}{a^\circ_{24}})\varphi_2 - (\underset{6\frac{EI}{l}}{c_{21}} + \underset{3EI/l}{c^\circ_{24}})\psi + ql^2/12 = 0$$

$$4\frac{EI}{l} + \frac{3EI}{l} + \frac{3EI}{l} = 10\,EI/l \qquad \frac{6EI}{l} + 3EI/l = 9\frac{EI}{l}$$

$$\Rightarrow \boxed{10\frac{EI}{l}\varphi_2 - 9\frac{EI}{l}\psi + ql^2/12 = 0} \quad (1'')$$

$$(2'):\ M_{12} = a_{12}\cancel{\varphi_{12}} + b_{12}\varphi_{21} - c_{12}\psi + \overline{M}_{12}$$

$$= b_{12}\varphi_2 - c_{12}\psi - ql^2/12$$

$$(2') \Rightarrow (a_{21} + b_{12} - a^\circ_{24})\varphi_2 - (c_{21} + c_{12} + c^\circ_{24})\psi + \cancel{(ql^2/12 - ql^2/12)} + ql^2/12 = 0$$

$$\frac{EI}{l}(4 + 2 - 3) = 3\,EI/l \qquad \frac{EI}{l}(6 + 6 + 3) = 15\,EI/l$$

$$\Rightarrow \boxed{3\frac{EI}{l}\cdot\varphi_2 - 15\frac{EI}{l}\psi + \frac{ql^2}{12} = 0} \quad (2'')$$

$$\frac{EI}{l}\begin{bmatrix}10 & -9\\ -\cancel{3}\,3 & +15\end{bmatrix}\begin{bmatrix}\varphi_2\\ \psi\end{bmatrix} = -\frac{ql^2}{12}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\begin{bmatrix}10 & -9\\ -3 & 15\end{bmatrix}\begin{bmatrix}\varphi_2\\ \psi\end{bmatrix} = -\frac{ql^2}{12}\cdot\frac{l}{EI}\begin{bmatrix}1\\ -1\end{bmatrix} = -\frac{ql^3}{12EI}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\begin{bmatrix}\varphi_2\\ \psi\end{bmatrix} = \begin{bmatrix}-0{,}049\\ 0{,}0569\end{bmatrix} ql^3/12EI \qquad \boxed{M_1 = -0{,}12\,ql^2}$$

$$\Rightarrow M_{12} = b_{12}\varphi_2 - c_{12}\psi - ql^2/12 = \frac{2EI}{l}(-0{,}049)\frac{ql^3}{12EI} - \frac{6EI}{l}\left(0{,}0569\frac{ql^3}{12}\right) - \frac{ql^2}{12}$$

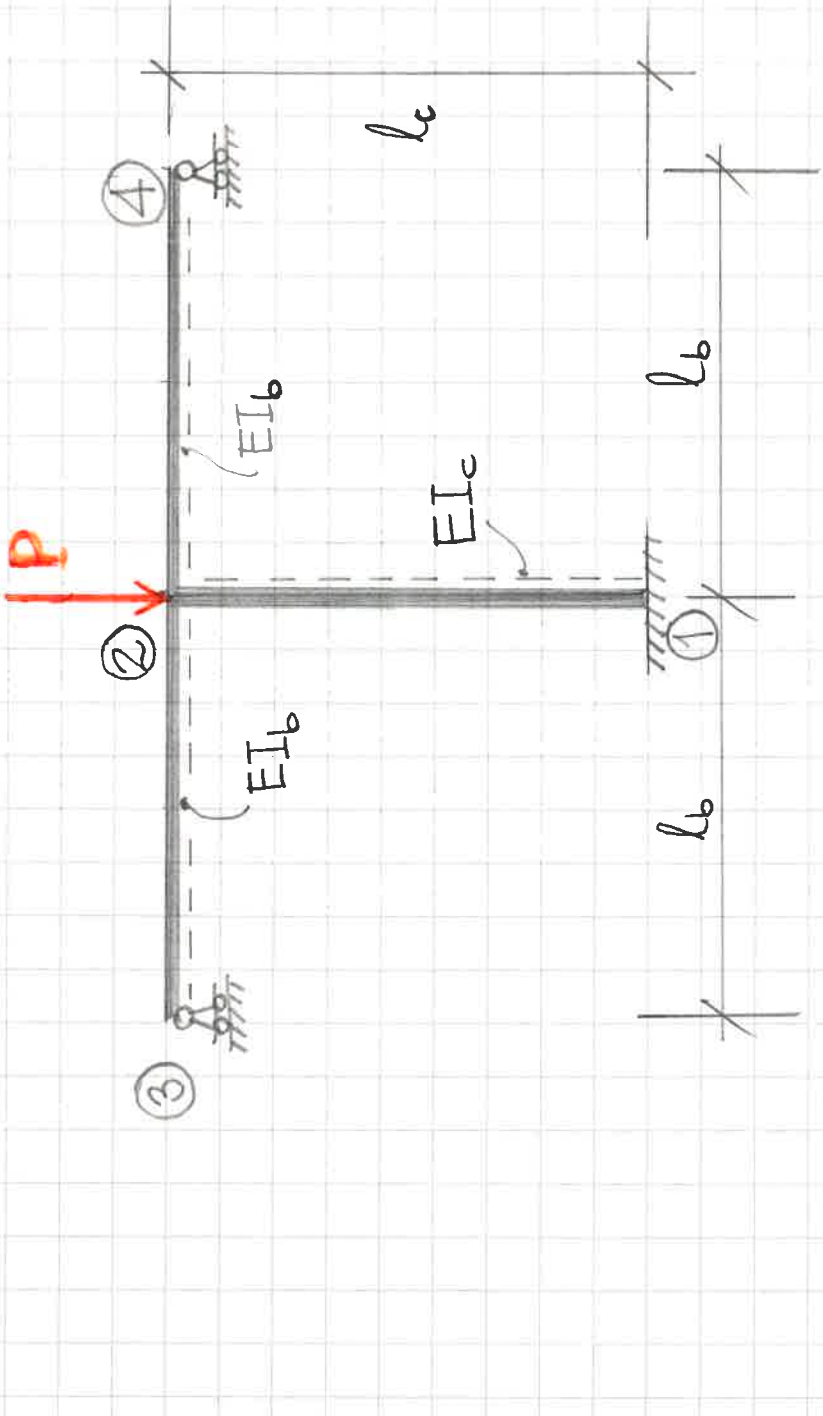
$$= -(0{,}0081 + 0{,}0072)ql^2 - ql^2/12 = -0{,}12\,ql^2 \equiv M_1$$

$$-(0{,}0081 + 0{,}0284)ql^2 - ql^2/12$$

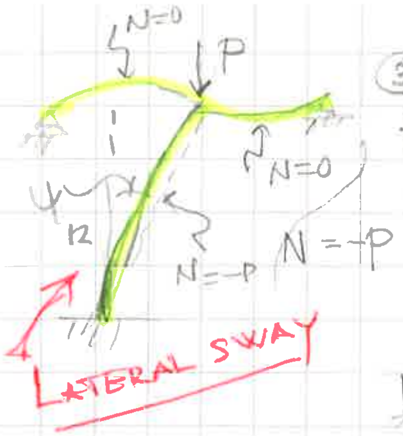
$M_{12} \equiv M_1$ ← bending moment

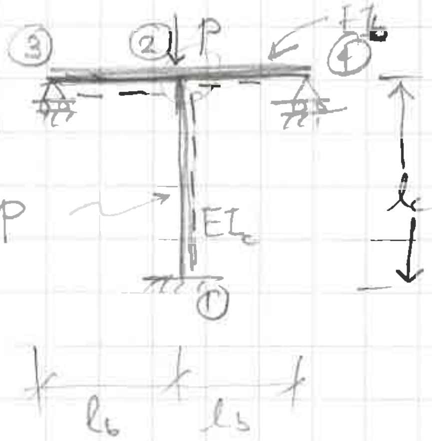
②

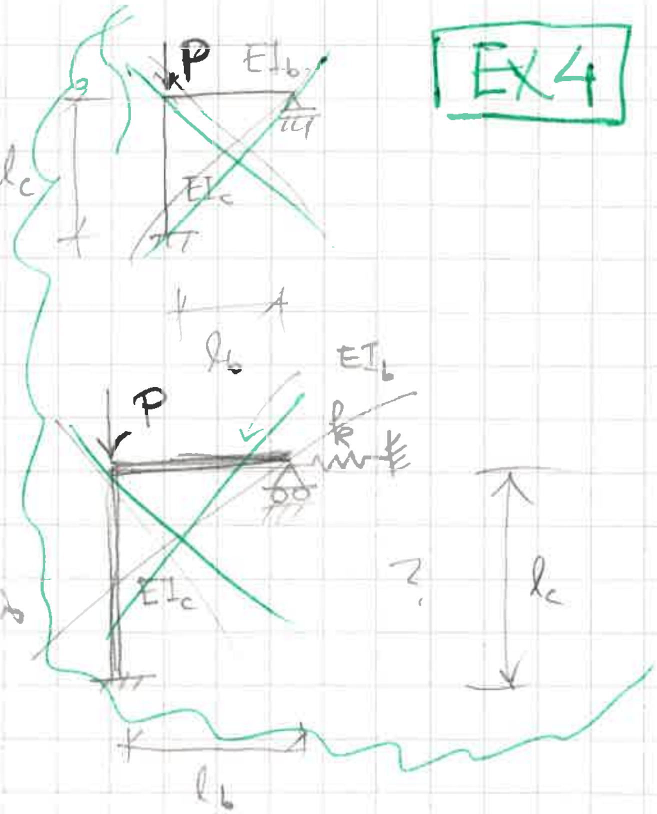
Ex 4



EX4





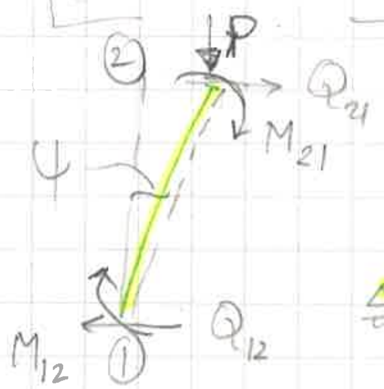


$$\boxed{\begin{aligned}\varphi_{23} = \varphi_{24} = \varphi_{21} &= \varphi_2 \\ \psi_{12} = \psi_{21} &= \psi\end{aligned}}\Bigg\}$$ needs two indep. equilibrium equations

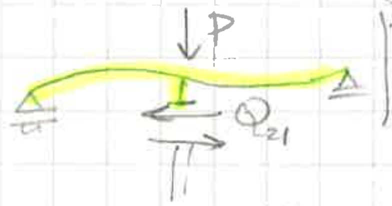
$$M^0_{23} + M_{21} + M^0_{24} = 0 \quad (1)$$

$$a^0_{23}\underbrace{\varphi_{23}}_{=\varphi_2} + a^0_{24}\underbrace{\varphi_{24}}_{=\varphi_2} + A_{21}\underbrace{\varphi_{21}}_{=\varphi_2} + B_{21}\cancel{\varphi_{12}}^{\,0} - C_{21}\psi_{21} = 0$$

$$\boxed{(a^0_{23} + a^0_{24} + A_{21})\,\varphi_2 - C_{21}\psi = 0} \quad (1')$$



$$Q_{21}\cdot l_c + M_{21} + M_{12} + Pl_c\cdot\psi = 0 \quad (2^*)$$



$$\boxed{Q_{21} = 0}$$

$$(2) \rightarrow M_{21} + M_{12} + Pl_c\cdot\psi = 0$$

$$(A_{21}\underbrace{\varphi_{21}}_{=\varphi_2} + B_{21}\cancel{\varphi_{12}}^{\,0} - C_{21}\psi) + (A_{12}\cancel{\varphi_{12}}^{\,0} + B_{12}\underbrace{\varphi_{21}}_{=\varphi_2} - C_{12}\psi) + P\cdot l_c\cdot\psi = 0$$

$$(A_{21} + B_{21})\,\varphi_2 - 2\cdot C_{12}\psi + P\cdot l_c\,\psi = 0 \quad (2')$$

$$\boxed{(A_{21} + B_{21})\,\varphi_2 - (2C_{12} + P\cdot l_c)\cdot\psi = 0} \quad (2'')$$

$$|N_{12}|$$

$$\begin{bmatrix} a^0_{23}+a^0_{24}+A_{21} & -C_{12} \\ \underbrace{A_{21}+B_{21}}_{=C_{12}} & -(2C_{12}+ \\ & +P\cdot l_c) \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rotation

$$, \quad k l_c = \lambda = \sqrt{\frac{|N_{12}|}{EI_c}}\, l_c$$

$$= \sqrt{\frac{P}{EI_c}} \cdot l_c = \lambda$$

$$\to P \cdot l_c^2 = \lambda^2 \cdot EI_c$$

($P > 0 \leftarrow$ compression)

$$\begin{bmatrix} a^0_{23}+a^0_{24}+A_{21} & -\overbrace{C_{12}}^{=A_{12}+B_{12}} \\ A_{21}+B_{21} & -\left(2\cdot C_{12}+\lambda^2\cdot \frac{EI_c}{l_c}\right) \end{bmatrix} = \underline{\underline{K}}(\lambda)$$

stiffness matrix

5 points

$$a^0_{23} = \frac{3EI_b}{l_b} = a^0_{24}$$

$$A_{21} = \frac{2\Psi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)} \cdot \frac{6EI_c}{l_c}$$

$$B_{21} = \frac{\phi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)} \cdot \frac{6EI_c}{l_c}$$

$$C_{12} = C_{21} = A_{12}+B_{12}$$

$$EI_c = 2EI_b = EI \qquad l_c = l_b$$

$$\downarrow \quad a^0_{23} = 3EI/l = a^0_{24}$$

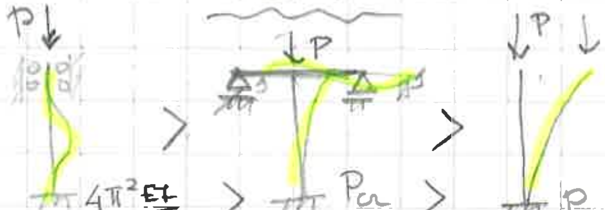
$$a^0_{23}+a^0_{24} = 6EI/l$$

$$\frac{EI}{l} \cdot \underbrace{\begin{bmatrix} 6+\frac{2\Psi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)}\cdot 6 & -\frac{2\Psi(\lambda)+\phi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)}\cdot 6 \\ \frac{2\Psi(\lambda)+\phi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)}\cdot 6 & -\left(2\cdot\frac{2\Psi(\lambda)+\phi(\lambda)}{4\Psi^2(\lambda)-\phi^2(\lambda)}+\lambda^2\right) \end{bmatrix}}_{\underline{\underline{K}}}$$

$$\det \underline{K}(\lambda) = 0 \to \lambda_{cr} \to P_{cr}.$$

Any engineer can bracket well the critical load.

$$\boxed{4\frac{\pi^2 EI}{l^2} < P_{cr} < \frac{\pi^2 EI}{4l^2}}$$



$$4\pi^2\frac{EI}{l^2} > P_{cr} > P_{cr} = \frac{\pi^2 EI}{4l}\Big/ l = \frac{\pi^2 EI}{4l^2}$$