CIV-E4010 Finite Element Methods in Civil Engineering Examination, April 6, 2020 / Niiranen

This examination consists of 3 problems rated by the standard scale 1...6.

Problem 1

Let us consider the finite element method in the context of structural mechanics and the theory of elasticity.

(i) Sketch the *linear* Lagrange-type *three-dimensional solid* finite element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of the local stiffness matrix and force vector of the element?

(ii) Sketch the *quadratic* Lagrange-type *Timoshenko beam* finite element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of the local stiffness matrix and force vector of the element?

(iii) Sketch the *quadratic* Lagrange-type degenerated *shell* element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of the local stiffness matrix and force vector of the element?

(iv) Describe, possibly with a few formula, how a finite element method (or software) forms the stress resultants of a shell element from the corresponding finite element approximations of the kinematic variables.

Problem 2

(i) Let us consider solving basic problems of *linear statics* in structural engineering (say, streching of a bar or bending of a beam, for simplicity) approximately by a finite element method. The resulting algebraic equation system can be written in the form $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$.

(1) Write down the equation system for the corresponding problem of *linear dynamics*. Explain briefly (2) the essential content and physical background of the system and (3) the main differences between solving these two types of equation systems (*statics* and *dynamics*).

(ii) (1) Write down the governing differential equation of the linear buckling problem of elastic columns. (2) Derive the corresponding weak form serving as a basis for the associated finite element formulation by assuming that the beam has simple supports at both ends.

(iii) Use two Hermite-type finite elements for finding an approximate solution to the problem of item (ii), i.e., for approximately determining the critical buckling load of the structure: (1) form the required finite element system equation; (2) form the characteristic equation of the problem.

Problem 3

The bilinear form of the variational formulation corresponding to the *Kirchhoff plate* bending problem, governed by the partial differential equation

$$\operatorname{div} \mathbf{div} \boldsymbol{M} = f \quad \text{in}\, \Omega,$$

can be written in the form

$$a(w, v) = \int_\Omega \boldsymbol{D}\boldsymbol{\kappa}(\nabla w) \cdot \boldsymbol{\kappa}(\nabla v) d\Omega,$$

$$\boldsymbol{D} = D \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix}, \quad D = \frac{Et^3}{12(1-\nu^2)},$$

$$\boldsymbol{\kappa}(\nabla w) = \begin{pmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ -2\partial^2 w/\partial x \partial x) \end{pmatrix}.$$

(i) Explain shortly, possibly with some basic mathematical notation, the meaning of both *conforming* and *nonconforming* finite element formulation in the context of this problem setting of structural meahanics.

(ii) For a conforming finite element method of the Kirchhoff plate problem, (with certain additional assumptions on the problem) the basic mathematical finite element error estimate is of the form

$$\|w - w_h\|_2 \leq Ch^{k-1}|w|_{k+1}.$$

(1) Define and name the quantities, variables, indeces and other notation appearing in the inequality, and (2) describe the information this estimate provides about the finite element method by referring to the notation defined.

(iii) By referring to the estimate in item (ii), describe which kind of error estimate one can write for the finite element approximation of the bending moment?

(iv) Let us solve the Kirchhoff plate problem with the classical nonconforming Morley element. Write down the constitutes for the stiffness matrix of the corresponding reference element.