CIV-E4010 Finite Element Methods in Civil Engineering Examination, May 25, 2020 / Niiranen

This examination consists of 3 problems rated by the standard scale 1...6. The course material can be independently utilized during the examination.

Problem 1

Let us consider the finite element method in the context of structural mechanics and the theory of elasticity.

(i) Sketch the *quadratic* Lagrange-type *three-dimensional hexahedral solid* finite element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of the local stiffness matrix and force vector of the element?

(ii) Sketch the *quadratic* Lagrange-type degenerated *quadrilateral shell* element: (1) identify the number of nodes in one element; (2) list the degrees of freedom present at each node; (3) determine the size of the local stiffness matrix and force vector of the element?

(iii) Explain briefly, possibly with a few formula, what is meant by numerical locking in the context of *Reissner–Mindlin plate* elements.

(iv) Describe, possibly with a few example formula, how a finite element method (or software) forms the stress resultants of a *Reissner–Mindlin plate* element from the corresponding finite element approximations of the kinematic variables.

Problem 2

(i) (1) Write down the governing differential equation of the linear buckling problem of elastic columns following the Euler–Bernoulli beam theory. (2) Derive the corresponding weak form, serving as a basis for the associated finite element formulation, by assuming that one end of the beam is clamped, while the other end is loaded by an axial compressive load.

(ii) Use two Hermite-type finite elements for finding an approximate solution to the problem of item (i), i.e., for approximately determining the critical buckling load of the structure: (1) form the required finite element system equation; (2) form the characteristic equation of the problem.

(iii) Let us consider solving the basic problems of *linear statics* of structural engineering (say, streching a bar or bending a beam, for simplicity) approximately by an appropriate finite element method. The resulting algebraic equation system can be written in the form $\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$.

(1) Write down the equation system for the corresponding problem of *geometrically nonlinear statics*. Explain briefly (2) the essential content and physico-mathematical background of the system and (3) the main differences between solving these two types of equation systems, *linear* and *nonlinear statics*.

Problem 3

The bilinear form of the variational formulation corresponding to the *Kirchhoff plate* bending problem, governed by the partial differential equation

$$\operatorname{div} \mathbf{div} \boldsymbol{M} = f \quad \text{in}\, \Omega,$$

can be written in the form

$$a(w, v) = \int_{\Omega} \boldsymbol{D} \boldsymbol{\kappa}(\nabla w) \cdot \boldsymbol{\kappa}(\nabla v) d\Omega,$$

$$\boldsymbol{D} = D \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix}, \quad D = \frac{Et^3}{12(1-\nu^2)},$$

$$\boldsymbol{\kappa}(\nabla w) = \begin{pmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ -2\partial^2 w/\partial x \partial x) \end{pmatrix}.$$

(i) Let us solve a Kirchhoff plate problem with the classical nonconforming Morley element. Write down the constitutes of (1) the element stiffness matrix and (2) force vector, without completing every little detail but providing the essential information and guidance for a fictional programmer.

(ii) For a conforming finite element method for the Kirchhoff plate problem, (with certain additional assumptions on the problem data) the basic mathematical finite element error estimate is of the form

$$\|w - w_h\|_2 \leq C h^{k-1} |w|_{k+1}.$$

(1) Define and name the quantities, variables, indeces and other notation appearing in the inequality, and (2) describe the information this estimate provides about the finite element method by referring to the notation defined.

(iii) By referring to the estimate in item (ii), describe and argue which kind of error estimate one can write for the finite element approximation of the bending moment tensor (or vector) or its components (i.e., bending and twisting moments)?