First course in probability and statistics
MS-A0501, MS-A0502, MS-A0503, MS-A0504, MS-A0509 — Exam
Department of Mathematics and Systems Analysis — 24.2.2021
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Instructions: Write briefly and clearly, but give reasons for your answers. A number only as an answer does not yield points. The exam has 4 problems, each worth 0–6 points.

Write your answers in clear handwriting, either on paper or a tablet computer, and send them in PDF form to the return box on the exam page. Make sure your answers contain: course code, last name, first name, student number and date.

At the end of the exam sheet there is a table for the CDF of the standard normal distribution.

T1 Peter and Lotta are playing the following game. They contribute (possibly inequal) sums of money to a prize pot. They roll an ordinary 6-sided die. If the result is 1 or 2, Lotta wins that round. If the result is 3, 4, 5 or 6, Peter wins that round. The first player to win two rounds wins the whole game and collects the prize pot.

(a) From the rules of the game it follows that there will be either 2 or 3 rounds. Calculate the expected value of the number of rounds. **(2p)**

(b) Peter contributes 20 euro to the pot. How much must Lotta contribute so that the game is fair, in other words, each player contributes an amount proportional to his or her probability of winning the game? **(2p)**

(c) Lotta won the first round. The dog Dot takes the die away and the game cannot be continued. The players decide to share the pot so that each player receives an amount proportional to his or her probability of winning the game, if the game could be continued. Calculate how the pot is shared. **(2p)**

Historical footnote. This problem takes us to the roots of probability theory. In the 17th century a Frenchman, de Méré, proposed a problem similar to this. While solving the problem, Blaise Pascal and Pierre de Fermat laid the foundations for the mathematical treatment of probability.

T2 The engine of a spaceship runs for a time T (seconds) and it causes the acceleration $a = 2\ \mathrm{m/s^2}$. The distance travelled during acceleration is $X = \frac{1}{2}aT^2$. The time T is a continuous random variable uniformly distributed over interval $I = [100,\ 150{+}r]$, where r is the **first digit** of your student number (one of $0, \dots, 9$, ignore possible letters). Write the value of r clearly in your answer and use this value in your calculations.

(a) Calculate the expected value of the distance travelled. **(2p)**

(b) Calculate the probability that the distance is more than 20 000 meters. **(2p)**

(c) Calculate the standard deviation of the distance. **(1p)**

(d) Ten similar spaceships are sent, and their engines are run for times $T_1, \dots, T_{10}$, which are independent and uniformly distributed over I. Calculate the probability that exactly three of the ships travel more than 20 000 meters during the acceleration. **(1p)**

T3 Let s be the **last digit** of your student number (one of $0, \dots, 9$, ignore possible letters). Write the value of r clearly in your answer and use this value in your calculations.

In a bag there are 100 coins, of which 98 are fair (with heads probability 0.5). One of the coins has heads probability 0.6 and one has 0.4. A coin is picked from the bag at random, and its heads probability is an unknown parameter Θ.

(a) What is the probability of obtaining only heads, when this coin is tossed $30 + s$ times? (The result must be a number, not an expression involving Θ.) **(1p)**

(b) The coin has been tossed $30 + s$ times, and only heads were obtained. Calculate the posterior distribution of Θ. **(2p)**

(c) According to the observations in (b), calculate the mode of the posterior distribution (MAP estimate). **(1p)**

(d) According to the observations in (b), calculate the mean of the posterior distribution. **(1p)**

(e) If this same coin is tossed three more times, what is the probability of obtaining only heads? **(1p)**

T4 Chebyshev's inequality says that if the random variable X has mean $\mu_X \in \mathbb{R}$ and standard deviation $\sigma_X \in]0, \infty[$, and $k \geq 1$, then $\mathbb{P}(|X - \mu_X| \leq k \cdot \sigma_X) \geq 1 - \frac{1}{k^2}$. In the case $k = \sqrt{2}$ it says that

$$\mathbb{P}(|X - \mu_X| \leq \sqrt{2} \cdot \sigma_X) \geq 1 - \frac{1}{\sqrt{2}^2} = \frac{1}{2}. \qquad (*)$$

In each of the following three cases, show (without using Chebyshev's inequality) that the claim $(*)$ is true, by calculating $\mathbb{P}(|X - \mu_X| \leq \sqrt{2} \cdot \sigma_X)$ either exactly or with two decimals.

(a) X is the result from rolling an ordinary 6-sided die. **(2p)**

(b) $X \sim \text{Exp}(2)$, that is, X has the exponential distribution with rate parameter $\lambda = 2$. It is then also known that $\mathbb{E}(X) = \text{SD}(X) = 1/\lambda$. **(2p)**

(c) $X \sim N(\mu, \sigma^2) = N(3, 5^2)$, that is, X has the normal distribution with mean $\mu = 3$ and standard deviation $\sigma = 5$. **(2p)**

Normal distribution table

The table below contains numerical values of the standard normal cumulative distribution function

$$\Phi(x) = F_Z(x) \;=\; \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999