-Online exam
-Starting: Thursday 25.2.2021 at 15:00
-Ending: Friday 26.2.2021 at 15:00
-Solutions should be clearly handwritten or LaTeX-compiled
-Upload solutions as pdf, jpg or png files

1. a) Starting from the general transversality condition in optimal control,

$$[h_x(x^*(t_f), t_f) - p^*(t_f)]^T \delta x_f + [H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f)]\delta t_f = 0,$$

justify the optimal control end point conditions (free final state, free final time, and free final state and time) given in the appendix below.

b) Suppose the Hamiltonian does not explicitly depend on time, and suppose a fixed final time problem. Show that $(d/dt)H(x^*(t), u^*(t), p^*(t)) = 0$ for all t.

2. Explain or describe shortly the following concepts:
a) minimum-time problem (optimal control)
b) the set of reachable states $R(t)$
c) bang-bang control
d) singular interval
e) admissible control law in discrete stochastic optimal control
f) dynamic programming algorithm (discrete stochastic optimal control).

3. Let A, B, C, D be matrices with dimensions $5 \times 4, 4 \times 6, 6 \times 3$ and 3×3, respectively. Using dynamic programming, determine the multiplication sequence that minimizes the number of scalar multiplications in computing $ABCD$.

4. At time t, a factory can choose what part $u(t)$, $0 \leq u(t) \leq 1$ of its output $x(t)$ to use for investments. The other part is sold with a unit price of \$1. The rate of increase of output at time t equals the amount of investments made. The output initially is $x(0) = c$. What kind of investment plan should the factory have to maximize its total sales over a period $[0, T]$? How does the solution depend on T? For simplicity, disregard discounting.

Hint: you can differentiate the switching function to find out if there are singular intervals.

5. Robert Merton is teaching a class on optimal consumption and investment (see Fig. 1), and he has a problem for you to solve.

   Let us assume that at a given time t capital $K(t)$ can be spent or invested. Find the optimal spending plan $c(t)$ for the fixed time interval $0 \le t \le T$, when we maximize the cumulative discounted momentarily utilities of spending

$$\int_0^T e^{-rt} c^{\alpha}(t)\, dt,\ 0 < \alpha < 1,$$

   where $r > 0$ is a constant discount factor, and α describes the shape of the utility function. We assume that the the initial capital is $K_0 = K(0)$, and that $K(T) = 0$. The momentarily return on the capital is $iK(t)$, where $i > 0$ is the interest rate.

   Full points are given already for a solution, where the optimal spending and capital are solved, without solving the integration constants.

   Hint 1: you can get rid of the $c(t)$ and turn this into a Calculus of Variations problem.
   Hint 2: when solving the system equation you can use the attempt $K(t) = Ae^{\frac{i-r}{1-\alpha}t}$ to obtain the particular solution of the nonhomogeneous differential equation.

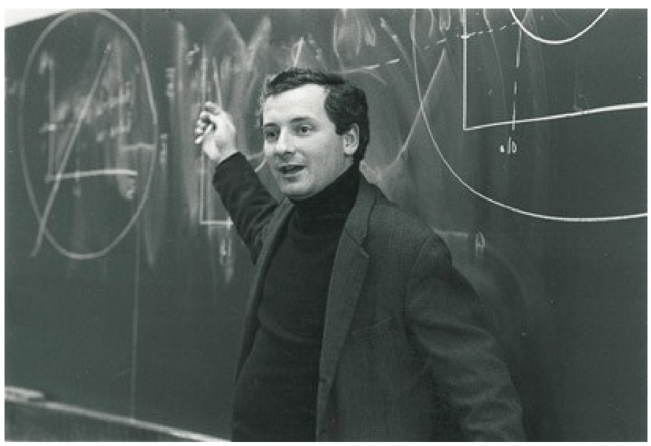

Figure 1: Robert Merton teaching a class.

APPENDIX:
HJB: $0 = J_t + \min_{u(t)} \left\{ g + J_x^T f \right\}$
E-L: $0 = g_x - \frac{d}{dt}(g_{\dot{x}})$
Hamiltonian: $H = g + p^T(t) f(x(t), u(t), t)$
costate: $\dot{p}(t) = -\frac{\partial H}{\partial x}$
free final state: $0 = g_{\dot{x}}$ or $h_x - p = 0$
free final time: $0 = g - g_{\dot{x}}\dot{x}$ or $H + h_t = 0$
free final state and time: $g_x = g_{\dot{x}} = 0$ or $h_x - p = 0 = H + h_t$
goal: $0 = g + \left[\frac{\partial g}{\partial \dot{x}}\right]^T \left[\frac{d\theta}{dt} - \dot{x}\right]$ or $H + h_t + (h_x - p)^T \frac{d\theta}{dt} = 0$
W-E: $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ continuous