-Online exam
-Starting: Thursday 25.2.2021 at 15:00
-Ending: Friday 26.2.2021 at 15:00
-Solutions should be clearly handwritten or LaTeX-compiled
-Upload solutions as pdf, jpg or png files

1. A machine is used for a time period of T to produce a good, after which the machine is sold. During its usage period, the machine can be maintained $u(t) \in [0, U]$, and then its resale value $x(t)$ remains higher than if it were not maintained. The amount of goods produced with the machine is proportional to the resale value of the machine. Let us assume that resale value of the machine is described by

$$\dot{x} = -d(t) + g(t)u(t),$$

where $d(t)$ is the decay of resale value ($d(t)$ is a decreasing function) caused by use of the machine, and $g(t)$ describes the efficiency of the maintenance, for which it holds $-d(t) + g(t)U \leq 0$, i.e., the machine can't be made better than it was before. The functional to maximize is then

$$J = e^{-rT}x(T) + \int_0^T e^{-rt}\left[\pi x(t) - u(t)\right]\, dt,$$

where the term $\pi x(t)$ is the profit flow at time t. Also, the discount factor $0 < r < 1$ is a constant.

Solve the optimal control w.r.t. the switching function. Can there be singular intervals?

2. Solve the optimal control for the problem

$$\min \int_0^1 [x(t) - u(t)],$$
$$s.t.\, \dot{x} = 1 + [u(t)]^2,$$

and $x(0) = 1$ and $x(1) = 1$.

3. Explain or describe shortly the following concepts:
a) autonomous problem in the calculus of variation
b) switching function
c) stochastic discrete time dynamic system
d) additive cost functional
e) the principle of optimality; give an example
f) cost-to-go at stage i and state x_i (stochastic optimal control)

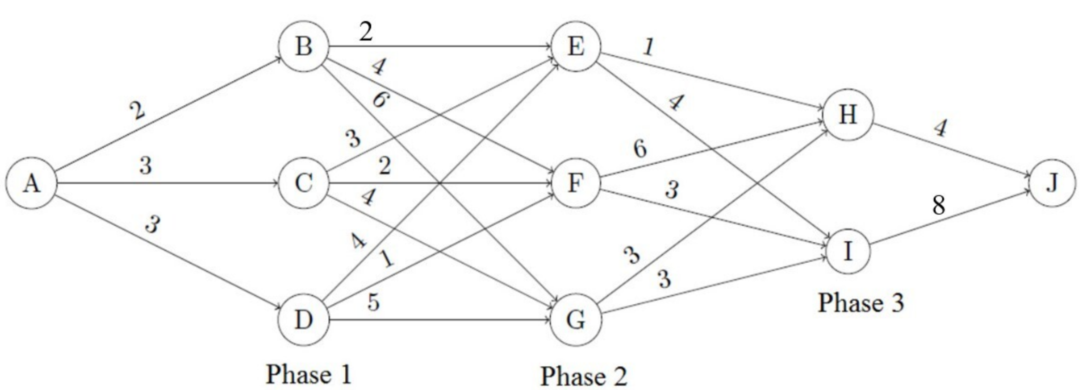
4. Suppose that we wish to know which stories in a H-story building are safe to drop your eggs from, and which will cause the eggs to break on landing (see Fig. 1). We make a few assumptions:

   - An egg that survives a fall can be used again.
   - A broken egg must be discarded.
   - The effect of a fall is the same for all eggs.
   - If an egg breaks when dropped, then it would break if dropped from a higher window.
   - If an egg survives a fall, then it would survive a shorter fall.

   Suppose N eggs are available, and you want to calculate the minimum number of trials needed in the *worst-case scenario* to solve which floors are safe to drop eggs from. Derive and carefully explain the DP algorithm you would use to solve the problem.

Figure 1: The tall building you make your experiment from.

5. The graph below depicts a flexible production line. The initial product A has to go through three phases to become the final product J. When the product goes from one node to another, the value, or the profit the product will give, is increased. The profit from each node to another is given next to the corresponding arc.



   Calculate the optimal cost-to-go function for each node and determine the steps that have to be taken for to *maximize* the profit.

APPENDIX:

HJB: $0 = J_t + \min_{u(t)} \left\{ g + J_x^T f \right\}$

E-L: $0 = g_x - \frac{d}{dt}(g_{\dot{x}})$

Hamiltonian: $H = g + p^T(t) f(x(t), u(t), t)$

costate: $\dot{p}(t) = -\frac{\partial H}{\partial x}$

free final state: $0 = g_{\dot{x}}$ or $h_x - p = 0$

free final time: $0 = g - g_{\dot{x}}\dot{x}$ or $H + h_t = 0$

free final state and time: $g_x = g_{\dot{x}} = 0$ or $h_x - p = 0 = H + h_t$

goal: $0 = g + \left[\frac{\partial g}{\partial \dot{x}}\right]^T \left[\frac{d\theta}{dt} - \dot{x}\right]$ or $H + h_t + (h_x - p)^T \frac{d\theta}{dt} = 0$

W-E: $g_{\dot{x}}$ and $g - g_{\dot{x}}\dot{x}$ continuous