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MS-C1081: Abstract Algebra
Final Exam

(1) **(10 points)** Let $\varphi : G \to H$ be a group homomorphism. Show that if $\gcd(|G|, |H|) = 1$ then $\varphi(g) = e_H$ for all $g \in G$.

(2) **(10 points)** Let G be a finitely generated proper subgroup of $(\mathbb{Q}, +)$. Prove or disprove: $\mathbb{Q}/G \cong \mathbb{Q}$.

(3) **(10+10 points)** Let G be a group. For any $x \in G$ consider the map $\varphi_x : G \to G$, $g \mapsto xgx^{-1}$, and denote $I = \{\varphi_x \mid x \in G\}$. Show that

(a) $I \trianglelefteq \mathrm{Aut}(G)$.

(b) $G/Z(G) \cong I$.

(4) **(5+10 points)** Let $K \leq H \leq G$ be groups, and assume that $\sigma(K) = K$ for all $\sigma \in \mathrm{Aut}(H)$. Show that

(a) $K \trianglelefteq H$.

(b) if $H \trianglelefteq G$ then $K \trianglelefteq G$.

Hint: Working with automorphisms introduced in Exercise (3) helps (in both cases).

(5) **(10 points)** Prove or disprove: every prime ideal of an integral domain is maximal.

(6) **(5+5+5 points)** Show that the following polynomials are irreducible in the indicated polynomial ring.

(a) $x^{2023} + 43x^{2022} + 2021$ in $\mathbb{Z}[x]$.

(b) $x^3 + 2x^2 + 3x - 1$ in $\mathbb{Z}[x]$.

(c) $x^4 + x + 1$ in $\mathbb{Z}_2[x]$.

(7) **(10+10 points)** By Exercise 6(c), $p(x) = x^4 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$ and thus $F = \mathbb{Z}_2[x]/(p(x))$ is a field of size 16. Put $\alpha = \overline{x}$.

(a) Compute the order of α^4 and α^5 in the multiplicative group F^*.

(b) Write α^6 and α^{13} as polynomials in α of degree at most 3.