Aalto University
School of Science

PHYS-E0420 Many-Body Quantum Mechanics
Second Exam 12.04.2020

General advice: You are allowed to use material (lecture noes, internet, books...), but collaboration is not permitted. Course results will be scaled in the end so that you will receive in the average similar grades the students of previous years got earlier. Return your solutions as a single clearly readable pdf-file by tomorrow at 9 am.

1. Write a short essay (about one page) on reduced density matrix. Try to answer such questions as: What is it, when is it used, when is it useful, how is it used?

2. Pure dilute Bose-Einstein condensate (with contact interactions) could be described be a Gross-Pitaevskii equation for its wavefunction $\psi(x)$

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) + V_T(\mathbf{x})\psi(\mathbf{x}) + \frac{4\pi\hbar^2 a_s}{m}|\psi(\mathbf{x})|^2\psi(\mathbf{x}) = \mu\psi(\mathbf{x}). \tag{1}$$

   (We choose the normalization $\int d\mathbf{x}|\psi(\mathbf{x})|^2 = \int d\mathbf{x} n(\mathbf{x}) = N$

   a) For a condensate without a trapping potential, what is the condensate wavefunction and its chemical potential?

   b) For a one-dimensional problem there is a (dark)-soliton solution, when $a_s > 0$. It has a density dip in the center and looks like

$$\psi(\mathbf{x}) = \sqrt{n}\tanh x/\xi, \tag{2}$$

   where $\tanh x$ is the hyberbolic tangent. Determine the so-called healing length ξ which characterizes the length scale for the variation in condensate wavefunction.

   Tips:

   - Asymptotically the condensate looks like a homogeneous condensate as in part a). This gives you insight on the chemical potential.
   - $\frac{d\tanh x}{dx} = 1/\cosh^2 x$
   - $\cosh^2 x - \sinh^2 x = 1$

3. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

$$H = -J\sum_{<i,j>} a_i^\dagger a_j + \frac{U}{2}\sum_i a_i^\dagger a_i^\dagger a_i a_i$$

   together with the Gutzwiller mean-field ansatz

$$|\Psi_{MF}\rangle = \prod_{i=1}^{M}\left[\sum_{n_i=0}^{\infty} f_{n_i}^{(i)}|n_i\rangle\right].$$

   a) Calculate the following quantities:

$$\langle\Psi_{MF}|a_i|\Psi_{MF}\rangle,$$
$$\Delta n^2 = \langle\Psi_{MF}|\hat{n}_i^2|\Psi_{MF}\rangle - \langle\Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle^2.$$

   (We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

b) Say we have two types of bosons (A and B) in our lattice and Hamiltonian is

$$H = -J_A \sum_{<i,j>} a_i^\dagger a_j - J_B \sum_{<i,j>} b_i^\dagger b_j + \frac{U_{AA}}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i + \frac{U_{BB}}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i + U_{AB} \sum_i a_i^\dagger a_i b_i^\dagger b_i$$

If you were to apply Gutzwiller type approach (with no correlations between sites) to this problem, what kind of mean-field ansatz would you try?

4. Consider the mean-field BCS Hamiltonian

$$H = \sum_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix}. \tag{3}$$

Diagonalize this by using the Bogoliubov transformation. Calculate the eigenenergies. You do not need to calculate the eigenvector amplitudes, we give them here:

$$u_{\mathbf{k}} = u_{\mathbf{k}\uparrow} = v_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}, \tag{4}$$

$$v_{\mathbf{k}} = v_{\mathbf{k}\uparrow} = -u_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}, \tag{5}$$

while the matrix for eigenvectors is

$$U = \begin{pmatrix} u_k & -v_k \\ v_k & u_k \end{pmatrix}. \tag{6}$$

How are the new quasiparticle operators $\gamma_{\mathbf{k}\uparrow}$, $\gamma_{\mathbf{k}\downarrow}$ related to the original operators $c_{\mathbf{k}\uparrow}$, $c_{\mathbf{k}\downarrow}$? Show that the quasiparticle operators obey the same (anti)-commutation relations as the original operators.