MS-C2111 Stochastic Processes
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2 June 2021
Exam

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This is a remote exam to be done independently. **Team work is strictly forbidden**. Detailed instructions are available in MyCourses. The exam contains 4 problems each worth 6 points.

1. Let $(X_0, X_1, X_2, \dots)$ be a discrete-time Markov chain in state space $\{-1, 0, +1\}$ with initial distribution $\pi = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ and transition matrix

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

having rows and columns indexed using $-1, 0, +1$. Which of the following are Markov chains? For those which are, write down the state space and transition matrix. For those which are not, explain carefully why.

(a) $A_t = X_{2t}$, **(2 p)**

(b) $B_t = 2X_t^3$. **(2 p)**

(c) $C_t = 3X_t^2$. **(2 p)**

2. Two computer algorithms called muGo and nuGo compete in a go tournament where the winner is declared to be the one who first wins two consecutive games in a row. Due to more powerful computing hardware, muGo is expected to win each game with probability $p = 0.75$, independently of other games. The state of the tournament after t rounds is denoted by (M_t, N_t) where M_t equals the number of consecutive games won by muGo, and N_t is the corresponding number for nuGo.

(a) Model the state of the tournament as a Markov chain on a finite state space, draw its transition diagram, and write down its transition matrix. **(2 p)**

(b) What is the probability that muGo wins the tournament? **(2 p)**

(c) Does the Markov chain have an invariant distribution? If yes, how many? **(2 p)**

3. Cars of type $i = 1, 2$ arrive to a one-way street at random time instants with expected interarrival times of ℓ_i minutes ($\ell_1 = 3$, $\ell_2 = 10$). The street has one parking space. If an arriving car finds the space vacant, the car parks there immediately. Otherwise the car drives away. A car of type i remains parked for an expected duration of m_i minutes ($m_1 = 5$, $m_2 = 20$). The interarrival times and parking times of type-i cars are mutually independent and exponentially distributed. In addition, type-1 cars behave independently of type-2 cars. Currently the parking space is vacant.

(a) Model the state of the parking space as a continuous-time Markov chain. Draw the transition diagram and write down the generator matrix of the chain. **(2 p)**

(b) Determine the probability that the parking space is vacant in steady state. **(2 p)**

(c) Explain how you can compute with the help of a computer the probability that the parking space is vacant after 30 minutes from now. **(1 p)**

(d) What is the probability that at least 2 cars arrive to the street during the 5-minute time interval starting after one hour? **(1 p)**

4. In a market dominated by two companies, the relative market share of company A is denoted by $X_t = \frac{M_t}{M_t + N_t}$ where M_t and N_t denote the subscriber counts of company A and company B in the end of week t. In the beginning, company A has $M_0 = 7$ subscribers and company B has $N_0 = 6$ subscribers. During each week $t = 1, 2, \ldots$ one new person arrives and becomes a subscriber of company A with probability X_{t-1} and a subscriber of company B otherwise.

(a) Is (X_t) a martingale? Prove your answer by a rigorous mathematical reasoning. **(3 p)**

(b) What is the expected relative market share of company A in the end of week 2? **(1 p)**

(c) Prove that the probability that the relative market share of company A ever reaches level 0.9 is at most 0.6. **(2 p)**