Aalto University
Department of Mathematics and Systems Analysis

MS-C1342 Linear Algebra (Noferini)

Exam 4.6.2021, 13:00–17:00.

Instructions: Answer as many questions as possible. For the exam mark, problems (labelled with numbers) are weighted proportionally to the amount of subquestions (labelled with letters) that they contain. Each subquestion has equal weight.

1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \pi \\ 1 \\ 1 \end{bmatrix}$$

   (a) Say whether the following claim is true or false: $A = A \cdot I_2$ is a QR decomposition of A. Justify your answer.

   (b) Using part (a) or otherwise, describe the set of solutions to the least squares problem $\min_{\mathbf{x}\in\mathbb{R}^2} \|A\mathbf{x} - \mathbf{b}\|_2$.

2. (a) Let the function $A \mapsto \|A\|$ be a matrix norm on $\mathbb{R}^{n\times n}$ and let $c_n > 0$ be a positive real number that may possibly depend on n, but not on the matrix A. Prove that the function $A \mapsto c_n\|A\|$ is also a matrix norm on $\mathbb{R}^{n\times n}$.

   (b) Using part (a) or otherwise, show that

$$\|A\|_{\max} = \max_{i,j=1}^{n} |A_{ij}|, \qquad \|A\|_\mu = n\|A\|_{\max}.$$

   are both matrix norms on $\mathbb{R}^{n\times n}$. Moreover, explain what is the relation between the condition number of a square matrix A associated with the norm $\|\cdot\|_\mu$ and the condition number of the same matrix relative to the norm $\|\cdot\|_{\max}$

   (c) A matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ is called *submultiplicative* if $\|AB\| \le \|A\|\|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$. Prove that $\|\cdot\|_\mu$ is submultiplicative but $\|\cdot\|_{\max}$ is not. *Hint. You may find it useful to first prove the following: if $v, w \in \mathbb{R}^n$ then $|v^T w| \le \|v\|_1\|w\|_\infty \le n\|v\|_\infty\|w\|_\infty$.*

3. Let $y \in \mathbb{R}$ be a real parameter and

$$B = \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix}.$$

   (a) Prove that the eigenvalues of B are $y + 1$ and $1 - y$ and compute an eigenvector associated with each of them. Then, say for which values of y the matrix B is symmetric positive definite and justify your answer.

   (b) Say for which values of y the linear system $B\mathbf{x} = \mathbf{b}$ does *not* have a unique solution for any possible right hand side $\mathbf{b} \in \mathbb{R}^{2\times 2}$. For each of these values, say whether it is true or false that the null space of B is the orthogonal complement (with respect to the dot product) of the range of B, justifying your answer.

   (c) Prove that

$$e^B = e \begin{bmatrix} \cosh(y) & \sinh(y) \\ \sinh(y) & \cosh(y) \end{bmatrix}$$

   where

$$\cosh(y) = \frac{e^y + e^{-y}}{2}, \qquad \sinh(y) = \frac{e^y - e^{-y}}{2}.$$

4. (a) A complex square matrix $C \in \mathbb{C}^{n\times n}$ is called *normal* if $CC^* = C^*C$. Prove the following generalization of the spectral theorem: a matrix is normal if and only if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal (but, in general, not necessarily real) matrix $\Lambda \in \mathbb{C}^{n\times n}$ such that $C = U\Lambda U^*$. *Hint. You may find it useful to recall the Schur decomposition theorem as discussed in the lectures.*

   (b) Explain why, if C is normal and $C = U\Lambda U^*$ is a decomposition as in part (a), then the diagonal entries of Λ are the eigenvalues of C. Then, using part (a) or otherwise, prove that if C is normal then

$$\|C\|_2 = |\lambda_{\max}(C)|$$

   where $\lambda_{\max}(C)$ is the eigenvalue of C having the largest modulus.

   (c) Give a counterexample of a square matrix D such that $\|D\|_2 \neq |\lambda_{\max}(D)|$, justifying your reasoning. Then, verify that for such a matrix $DD^* \neq D^*D$.