ELEC-C9420 Introduction to Quantum Technology, Fall 20
Midterm exam 1, part B, 22.10.2020

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Problem B1

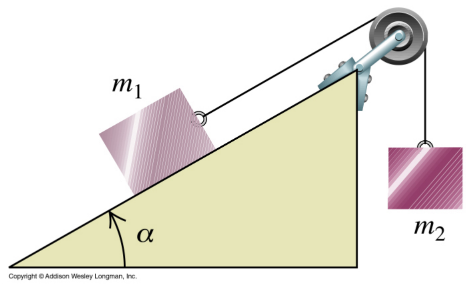

Figure 1: Setup of Problem B1

Two boxes of masses m_1 and m_2 are attached by a strong line passing through a pulley without slipping as in Figure 1. In addition to the masses of the boxes, the pulley wheel also has mass m_p, and is of an approximately cylindrical shape with radius R. The boxes are affected by the gravitational acceleration g. The mass of the line can be neglected.

a) Derive an expression for the inclination angle α of the ramp (in terms of the other given quantities) such that the system is in equilibrium.
b) The coefficient of kinetic friction between the box 1 and the ramp is μ. Solve for the acceleration of box 1 in the case that the system is not in equilibrium.

Problem B2

Two cars A and B (masses $m_A = 800$ kg and $m_B = 1300$ kg) collide with initial speeds $v_A = 70.0$ km/h and $v_B = 80.0$ km/h, so that the angle between their initial velocities is $\alpha = 70.0°$. In the collision, the cars get stuck together, and after the collision they slide together on the road. The coefficient of kinetic friction between the wreckage and the road is $\mu = 0.800$, and the gravitational acceleration $g = 9.81$ m/s^2. Assume that the cars move only in the horizontal plane.

a) How far from the point of collision do the cars slide?
b) How long does it take (in time) for the cars to stop after the collision?

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Midterm exam 1, part B, model solutions

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Problem B1

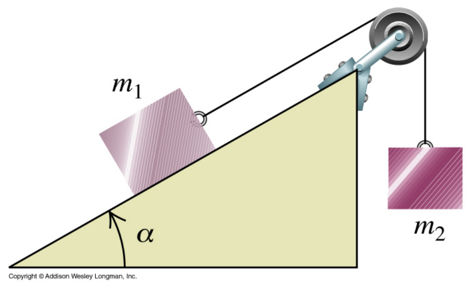

Figure 1: Setup of Problem B1

Two boxes of masses m_1 and m_2 are attached by a strong line passing through a pulley without slipping as in Figure 1. In addition to the masses of the boxes, the pulley wheel also has mass m_p, and is of an approximately cylindrical shape with radius R. The boxes are affected by the gravitational acceleration g. The mass of the line can be neglected.

a) Derive an expression for the inclination angle α of the ramp (in terms of the other given quantities) such that the system is in equilibrium. You may neglect frictional forces.
b) Now assume that the coefficient of kinetic friction between the box 1 and the ramp is μ_k. Solve for the acceleration of box 1 in the case that the system is not in equilibrium.

Solution:

a) Forces exerted on Box 1:

- weight $G_1 = m_1 g$,
- normal force of the ramp N,
- support force of the line T_1. (0.5p)

For Box 1, let's choose positive x-axis along the ramp upwards and positive y-axis perpendicular to the ramp. Then the components of the weight are $G_{1,x} = G_1 \sin\alpha$ and $G_{1,y} = G_1 \cos\alpha$. (0.5p) By Newton's 1st law, the equilibrium condition in the y-direction gives

$$N - G_{1,y} = 0 \quad \Rightarrow \quad N = G_{1,y} = m_1 g \cos\alpha\,. \quad \text{(0.5p)}$$

The equilibrium condition in x-direction gives

$$T_1 - G_{1,x} = 0 \quad \Rightarrow \quad T_1 = m_1 g \sin\alpha\,. \quad \text{(0.5p)}$$

The forces exerted on Box 2:

- weight $G_2 = m_2 g$,
- support force of the line T_2. (0.5p)

If we choose the positive direction upwards, the equilibrium condition for Box 2 reads

$$T_2 - G_2 = 0 \quad \Rightarrow \quad T_2 = m_2 g\,. \quad \text{(0.5p)}$$

The support forces T_1 and T_2 are equal in this case, because the line mediates the force without loss. (0.5p) Accordingly, we get

$$m_2 g = m_1 g \sin\alpha \quad \Rightarrow \quad \alpha = \arcsin\left(\frac{m_2}{m_1}\right)\,. \quad \text{(0.5p)}$$

b) We get for the equation of motion for Box 1

$$m_1 a_1 = T_1 - G_{1,x} \pm F_{\mu_s}\,. \quad \text{(0.5p)}$$

Here, the sign of the frictional force $F_{\mu_s} = \mu_s N = \mu_s m_1 g \cos\alpha$ depends on the direction of motion of the box, the kinetic friction being always to the opposite direction of motion. (0.5p) The equation of motion for Box 2 reads

$$m_2 a_2 = T_2 - G_2\,. \quad \text{(0.5p)}$$

Now, we must also take into account the rotational motion of the pulley wheel. For the pulley wheel we have the equation of motion

$$I\alpha = T_1 R - T_2 R\,, \quad \text{(0.5p)}$$

where I is the moment of inertia and α the angular acceleration of the wheel.

In this case, the two forces T_1 and T_2 are not the same, as the angular acceleration of the pulley wheel 'absorbs' some of the force. However, the accelerations of the two boxes are the same in magnitude, as they are connected by the line. (0.5p) Their directions are opposite, however, due to how we chose the directions of positive coordinate axes, so $a_1 = -a_2 = a$. Since the line does not slip on the pulley wheel, we also have $a_2 = R\alpha = -a$. (0.5p)

Solving for T_2 from the last equation we get

$$T_2 = T_1 - \frac{I\alpha}{R} = T_1 + \frac{Ia}{R^2}\,.$$

Substituting this into the equation of motion for Box 2 gives

$$-m_2 a = T_1 + \frac{Ia}{R^2} - G_2 \quad \Rightarrow \quad T_1 = -m_2 a - \frac{Ia}{R^2} + m_2 g\,.$$

Substitution to the equation of motion for Box 1 gives

$$m_1 a = -m_2 a - \frac{Ia}{R^2} + m_2 g - m_1 g \sin\alpha \pm F_{\mu_s}\,.$$

Solving for the accelation gives

$$a = \frac{m_2 g - m_1 g \sin\alpha \pm F_{\mu_s}}{m_1 + m_2 + \frac{I}{R^2}}\,.$$

Substituting $F_{\mu_s} = \mu_s m_1 g \cos\alpha$ and $I = \frac{1}{2} m_p R^2$ we find the expression

$$a = \frac{m_2 g - m_1 g \sin\alpha \pm \mu_s m_1 g \cos\alpha}{m_1 + m_2 + \frac{1}{2} m_p}\,. \quad \text{(0.5p)}$$

Here, the plus sign applies in the case when Box 1 is moving in the negative x-direction down the ramp, and the minus sign applies when Box 1 is moving to the positive x-direction up the ramp. (0.5p)

Problem B2

Two cars A and B (masses $m_A = 800$ kg and $m_B = 1300$ kg) collide with initial speeds $v_A = 70.0$ km/h and $v_B = 80.0$ km/h, so that the angle between their initial velocities is $\alpha = 70.0°$. In the collision, the cars get stuck together, and after the collision they slide together on the road. The coefficient of kinetic friction between the wreckage and the road is $\mu = 0.800$, and the gravitational acceleration $g = 9.81$ m/s^2. Assume that the cars move only in the horizontal plane.

a) How far from the point of collision do the cars slide?
b) How long does it take (in time) for the cars to stop after the collision?

Solution:
Let's first solve for the speed of the wreckage right after the collision. Choose the positive x-axis to the direction of Car A's initial velocity, and the y-axis perpendicularly to the x-axis, so that Car B approaches Car A from the negative y-direction in the beginning. Then the initial velocities of the two cars are

$$\vec{v}_A = v_A\hat{\imath}\,, \quad \vec{v}_B = v_B\cos\alpha\hat{\imath} + v_B\sin\alpha\hat{\jmath}\,.$$

Let's denote the speed of the cars immediately after the collision by u and the angle of their velocity with respect to the positive x-axis by β. The velocity of the wreckage immediately after the collision can then be expressed as $\vec{u} = u\cos\beta\hat{\imath} + u\sin\beta\hat{\jmath}$. The external forces on cars, their weights and the normal forces of the ground, cancel out during the collision, so the total momentum is conserved. (1p) We get from the conservation of the total momentum

$$\begin{aligned} &x\text{-component: } m_Av_A + m_Bv_B\cos\alpha = (m_A+m_B)u\cos\beta\,, \quad (1\text{p}) \\ &y\text{-component: } m_Bv_B\sin\alpha = (m_A+m_B)u\sin\beta\,. \quad (1\text{p}) \end{aligned}$$

From the second equation we may solve $u = \frac{m_Bv_B\sin\alpha}{(m_A+m_B)\sin\beta}$, and substituting into the first equation we get

$$m_Av_A + m_Bv_B\cos\alpha = \frac{m_Bv_B\sin\alpha}{\tan\beta} \quad\Rightarrow\quad \beta = \arctan\left(\frac{m_Bv_B\sin\alpha}{m_Av_A + m_Bv_B\cos\alpha}\right) \approx 46.8632°\,.$$

For the speed we get $u = \frac{m_Bv_B\sin\alpha}{(m_A+m_B)\sin\beta} \approx 63.7737$ km/h. (1p)

a) The wreckage before the slide has kinetic energy $K = \frac{1}{2}(m_A+m_B)u^2$. (0.5p) In this case the gravitational force and the normal force are equal but opposite, so the net force on the wreckage equals the friction force. (0.5p) The kinetic friction $F_\mu = \mu_k(m_A+m_B)g$ does work $W = -F_\mu d$ on the wreckage during the slide of distance d, which by the work-energy theorem must equal the change in kinetic energy $-K$. (0.5p) Thus, we get for the distance

$$d = \frac{K}{F_\mu} = \frac{\frac{1}{2}(m_A+m_B)u^2}{\mu_k(m_A+m_B)g} = \frac{u^2}{2\mu_kg} \approx 20.0\text{ m}\,. \quad (0.5\text{p})$$

b) The change in momentum of the wreckage during the slide $\Delta p = -(m_A+m_B)u$ is equal to the impulse given by the friction force $I = -F_\mu\Delta t$, where Δt is the duration of the slide. (1p) Therefore, we get

$$\Delta t = \frac{(m_A+m_B)u}{\mu_k(m_A+m_B)g} = \frac{u}{\mu_kg} \approx 2.26\text{ s}\,. \quad (1\text{p})$$