CIV-E1020 - Mechanics of Beam and Frame Structures –

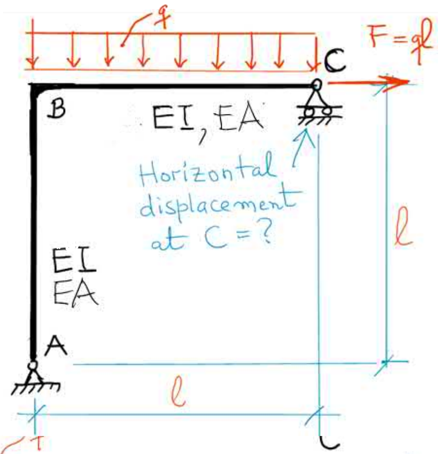
Duration: 3h + additional 2x15 min You should return scanned hand written answers as in a contact exam in a good pdf-fquality

It is compulsory to solve only THREE (3) EXERCISES that you choose freely: only three best exercises (answers) will be graded even if the student solves four.

1) Results given without shown the logical steps needed to achieve them will be ignored even if correct.
2) Sequentially number (numeroi juoksevasti) your answer papers 1(*n*) ... *n*(*n*), where *n* =total number of pages
3) Write readably your name, family name and student number.
4) Name the pdf-file: `Studdent_ID_Name_Date.pdf` Make a good quality scanned pdf
5) All additional material, like listings, graphs can be appended as pdfs to the answer
6) Please check the physical units of your answers.

Exam. points		Grade
≥13,75		5
12,75	13,5	4
9,75	12,5	3
8,25	9,5	2
7	8	1
<7		fail=0

Examination 26/10/2021

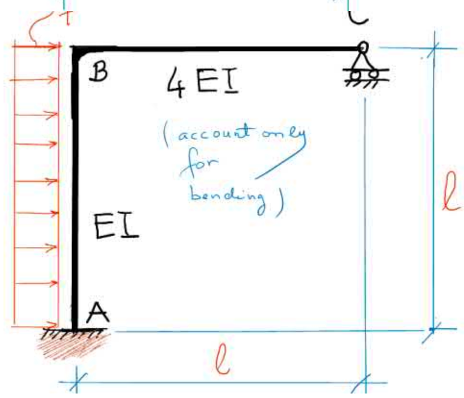


The material is linear elastic in all the structures below

1. **Use the dummy unit-load theorem** (or method) and determine the horizontal displacement at roller **C** [3 point]. Account for both the effects of bending and axial forces when computing displacement. Ignore the shear effects. [overall = 5 points]

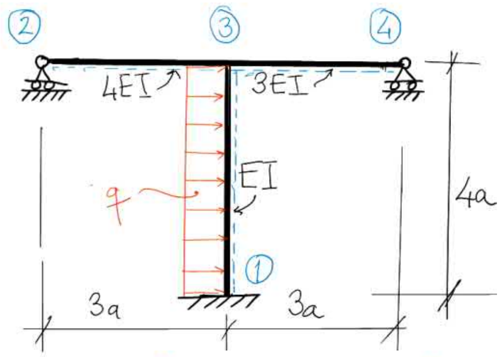
$$EI = \alpha EA \cdot l^2$$

Hints: is it statically determined? Determine support reactions
Then determine and draw accurately the bending moment [1 point] and axial force [1 point] diagrams



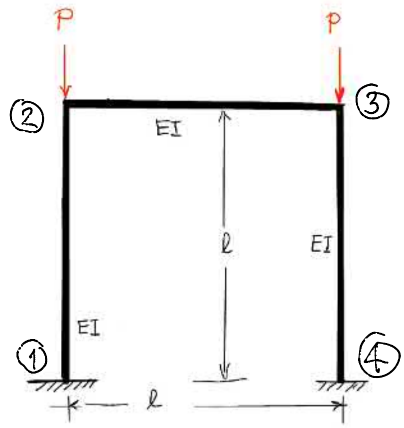
2. Use the general force method and
a) determine the bending moment at support *A* [3 points] and draw accurately the bending moment diagram [1 point]. *Account only for effects of bending*

b) Determine the support reaction at *C* (value and direction). [1 p].



3. Use Slope-Deflection Method and
a) **determine the bending moment** at clamping support *1* [4 points]
b) use results from question *a*) and determine the horizontal displacement at roller 4 [1 point] (all other methods are welcomed for evaluating the displacement)

Hint a): If you wish you can use the stiffness-moment relation for hinged beams where appropriate

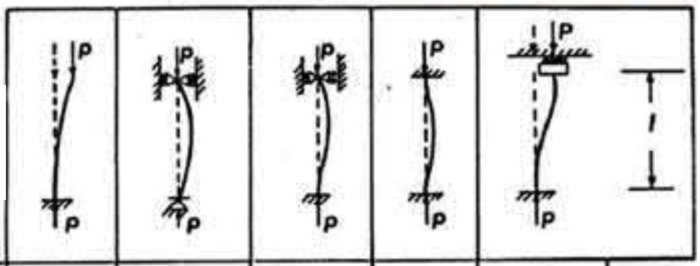


4. Buckling of sway-frames
The Frame is loaded symmetrically with by two concentrated loads *P*.
Use Slope-Deflection Method and 1) derive the explicit expression, in terms of Berry's stability functions, of the needed **criticality condition for determining the critical buckling load** *P* [3 points]. **Hint**: assume anti-symmetric buckling mode
2) **solve numerically for the value of the buckling load** *P* [1 point].
3) Give a **bracket** for the value of **buckling load** using cleverly the Euler's basic cases (see tables in the formulary) [1 point].

Euler's basic buckling cases

Eulerin perusnurjahdus

$$P_{cr} = \mu \frac{\pi^2 EI}{\ell^2}$$



Tapaus	1	2	3	4	5
μ	0.25	1	2.046	4	1

$$M_{ij} = A_{ij}\phi_{ij} + B_{ij}\phi_{ji} - C_{ij}\psi_{ij} + \bar{M}_{ij} \qquad M^0_{ij} = A^0_{ij}\varphi_{ij} - C^0_{ij}\psi_{ij} + \overline{M}^0_{ij}$$

Beam-column with constant flexural rigidity:

$$A_{ij} = A_{ji} = \frac{2\psi(kL)}{4\psi^2(kL) - \phi^2(kL)} \frac{6EI}{L} \qquad B_{ij} = B_{ji} = \frac{\phi(kL)}{4\psi^2(kL) - \phi^2(kL)} \frac{6EI}{L}$$

$$C_{ij} = A_{ij} + B_{ij} \qquad \underbrace{A^0_{ij} = C^0_{ij} = \frac{1}{\psi(kL)} \frac{3EI}{L},}_{\text{one end is hinged } B^0_{ij} = B^0_{ji} = 0} \qquad kL \equiv L\sqrt{\frac{P}{EI}}$$

Berry's functions:

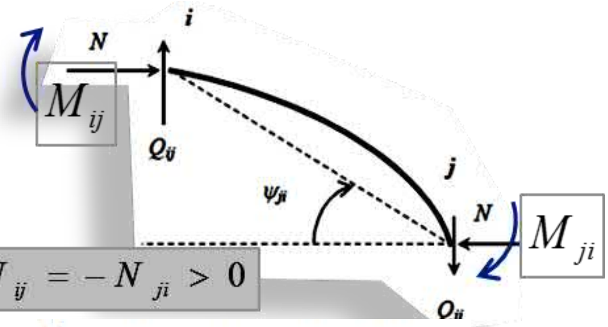
Olkoon $\lambda \equiv kL$, $\qquad \lambda \equiv kL$

Puristettu sauva:

Compression: $\phi(\lambda) = \frac{6}{\lambda}\left(\frac{1}{\sin\lambda} - \frac{1}{\lambda}\right)$, $\psi(\lambda) = \frac{3}{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\tan\lambda}\right)$ ja $\chi(\lambda) = \frac{24}{\lambda^3}\left(\tan\frac{\lambda}{2} - \frac{\lambda}{2}\right)$,

Vedetty sauva:

Extension: $\phi(\lambda) = \frac{6}{\lambda}\left(-\frac{1}{\sinh\lambda} + \frac{1}{\lambda}\right)$, $\psi(\lambda) = \frac{3}{\lambda}\left(-\frac{1}{\lambda} + \frac{1}{\tanh\lambda}\right)$ ja $\chi(\lambda) = \frac{24}{\lambda^3}\left(-\tanh\frac{\lambda}{2} + \frac{\lambda}{2}\right)$,



$$N \equiv -N_{ij} = -N_{ji} > 0$$

$$Q_{ij} = Q^0_{ij} - (M_{ij} + M_{ji})/L - N\psi_{ij}$$

$$\overline{M}_{12} \equiv MK_1$$

$$\overline{M}_{ij}, \overline{M}_{ji}$$

N:o	Kuormitus	Kiinnitysmomentit: MK_1, MK_2
1	q, L	$MK_1 = -\frac{qL^2}{12}$, $MK_2 = \frac{qL^2}{12}$
2		

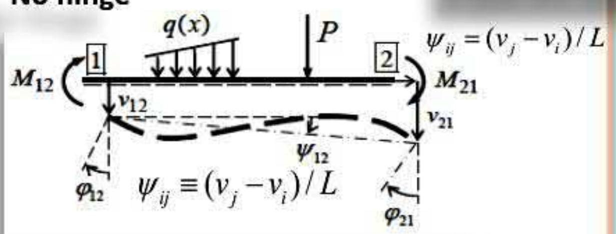
$$\overline{M}_{ij} \qquad \overline{M}_{ji}$$

N:o	Kuormitus	Nivel oikeassa päässä: MK_1^0	Nivel vasemmassa päässä: MK_2^0
1	q, L	$MK_1^0 = -\frac{qL^2}{8}$	$MK_2^0 = \frac{qL^2}{8}$

The stiffness equation relating the end-moments to the end-displacements

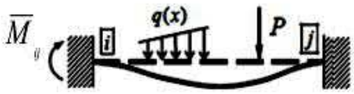
If you are using lecture's notations

No hinge



$$M_{ij} = a_{ij}\varphi_{ij} + b_{ij}\varphi_{ji} - c_{ij}\psi_{ij} + \overline{M}_{ij}, \quad i \neq j$$

$$a_{ij} = \frac{4EI}{L}, \quad b_{ij} = \frac{2EI}{L}, \quad c_{ij} = \frac{6EI}{L} \qquad (EI\text{-constant})$$



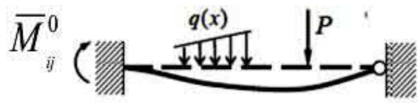
Fixed end-moment resulting from external mechanical loading, look from tables

One node is hinged

The is a superscript "0" means that the support at end *j* is hinged

$$M_{ij}^0 = a_{ij}^0\varphi_{ij} - c_{ij}^0\psi_{ij} + \overline{M}_{ij}^0$$

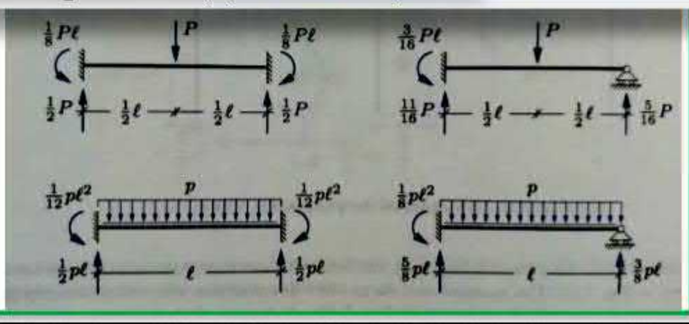
$$a_{12}^0 = c_{12}^0 = \frac{3EI}{L} \qquad \psi_{ij} = (v_j - v_i)/L$$



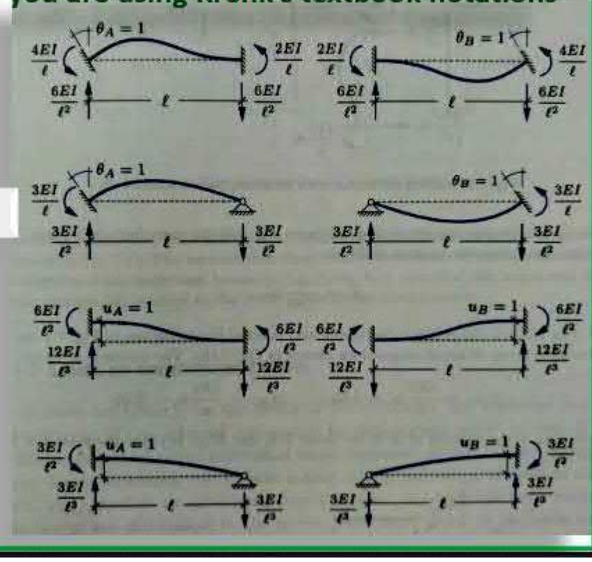
Fixed end-moment resulting from external mechanical loading, look from tables

If you are using Krenk's textbook notations

Bending Moments (pay attention to the sign convention to convert to Fixed-End-Moments)



If you are using Krenk's textbook notations



Maxwell-Mohr integrals table

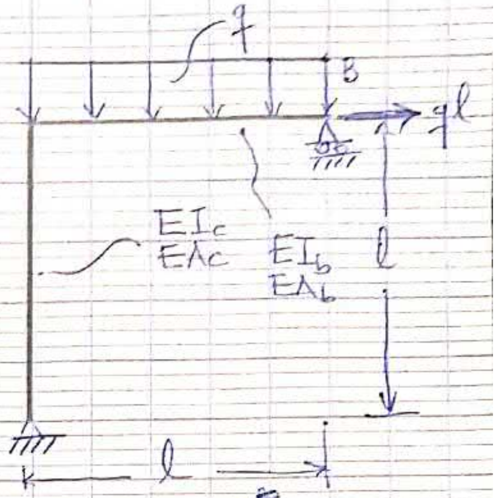
TABLEAU DES INTEGRALES $\int_0^{\ell} {}^{i}M\,{}^{k}M\,dx$

${}^{k}M$ \ ${}^{i}M$	c, ℓ	c, ℓ	d, ℓ	c, d, ℓ	$2^{ème}$ deg. c, ℓ
a, ℓ	$ac\ell$	$\frac{1}{2}ac\ell$	$\frac{1}{2}ad\ell$	$\frac{1}{2}a\ell(c+d)$	$\frac{2}{3}ac\ell$
a, ℓ	$\frac{1}{2}ac\ell$	$\frac{1}{3}ac\ell$	$\frac{1}{6}ad\ell$	$\frac{1}{6}a\ell(2c+d)$	$\frac{1}{3}ac\ell$
b, ℓ	$\frac{1}{2}bc\ell$	$\frac{1}{6}bc\ell$	$\frac{1}{3}bd\ell$	$\frac{1}{6}b\ell(c+2d)$	$\frac{1}{3}bc\ell$
a, b, ℓ	$\frac{1}{2}(a+b)c\ell$	$\frac{1}{6}(2a+b)c\ell$	$\frac{1}{6}(a+2b)d\ell$	$\frac{\ell}{6}[a(2c+d)+b(c+2d)]$	$\frac{1}{3}(a+b)c\ell$
$2^{ème}$ deg. a, ℓ	$\frac{1}{3}ac\ell$	$\frac{1}{4}ac\ell$	$\frac{1}{12}ad\ell$	$\frac{1}{12}a\ell(3c+d)$	$\frac{1}{5}ac\ell$
$2^{ème}$ deg. a, ℓ	$\frac{2}{3}ac\ell$	$\frac{5}{12}ac\ell$	$\frac{1}{4}ad\ell$	$\frac{1}{12}a\ell(5c+3d)$	$\frac{7}{15}ac\ell$

Ex. 1

Detailed solutions for the exam. 26/10/2021

- Support reactions



$\overset{\curvearrowleft}{A}$: $V_B \cdot \cancel{l} - ql \cdot \cancel{l} + ql \cdot \frac{l}{2} = 0$

$V_B = ql + ql/2 = \underline{3/2\, ql}$

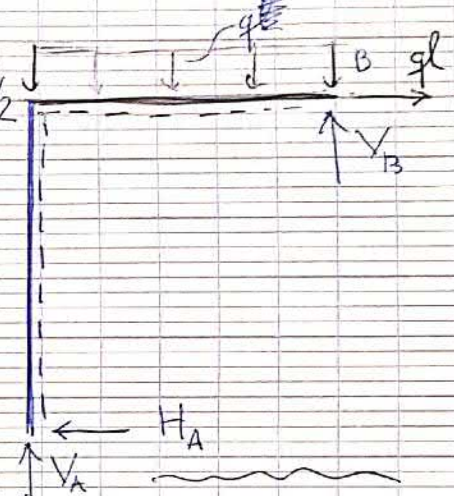
$\uparrow$: $V_A + V_B - ql = 0 \rightarrow V_A = ql - V_B =$

$V_A = ql - 3/2\, ql = \underline{-1/2\, ql}$

$\overset{\curvearrowleft}{B}$: $H_A \cdot \cancel{l} + V_A \cdot \cancel{l} - ql \cdot \frac{l}{2} = 0$

$H_A = -V_A + ql/2 = -(-1/2\, ql) + ql/2$

$= \underline{ql}$



Cross-check : $\leftarrow$? : $\underbrace{H_A}_{ql} - ql = 0$

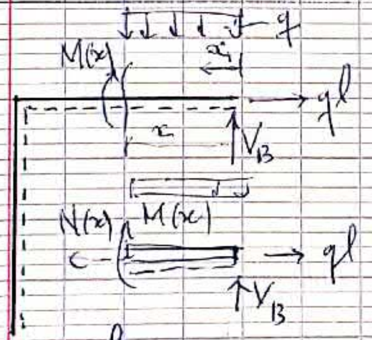
TRUE.

We can rely on the calculated support reactions.

$1 \cdot u_B = \sum \int_S \overline{M} \kappa \, dx + \sum \int_S \overline{N} \varepsilon_0 \, dx = \sum \int_S \frac{\overline{M} M}{EI} dx + \sum \int_S \frac{\overline{N} N}{EA} dx \quad (1)$

Sum and integrals over the whole structure.

$\rightarrow$ Therefore we need $M(x)$, $N(x)$ and $\overline{M}(x)$, $\overline{N}(x)$



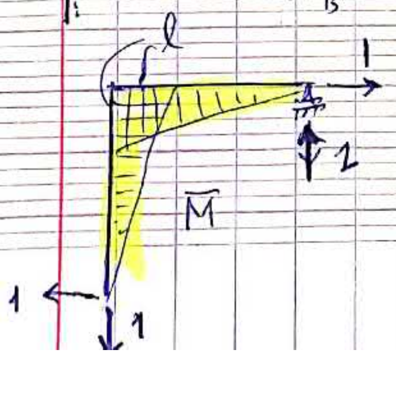
$M(x_1) + q \cdot x_1^2/2 - V_B \cdot x_1 = 0$

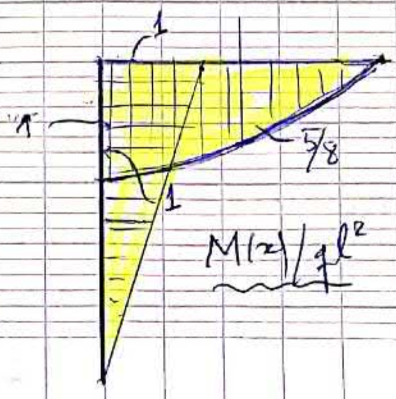
$M(x_1) = V_B \cdot x_1 - q x_1^2/2 = \frac{3}{2} ql\, x_1 - q \frac{x_1^2}{2}$

$M(x_1 = l/2) = \frac{3}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{4}$

$= -\frac{1}{8} + \frac{3 \cdot 2}{4 \cdot 2}$

$\frac{6}{8} - \frac{1}{8} = 5/8$





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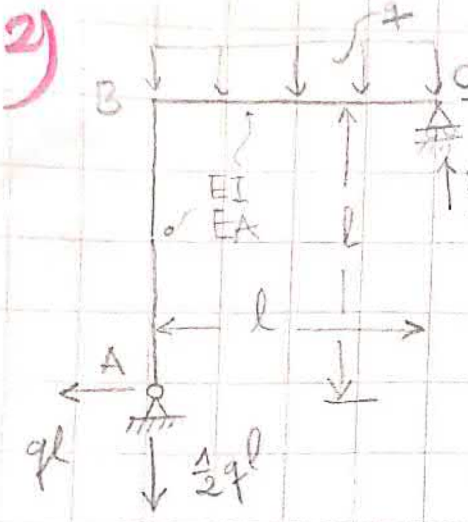
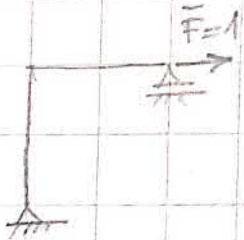
2)





EX1

Bending: u_B

Vereshagin integration rule, Simpson's rule

$$EI u_B = \frac{1}{2} ql^2 \cdot l \cdot \frac{2}{3} l + \frac{l}{6}\left[l \cdot ql^2 + 4 \cdot \frac{5}{8} ql^2 \cdot \frac{1}{2} l\right]$$

$1 + \frac{5}{4} = \frac{9}{4}$

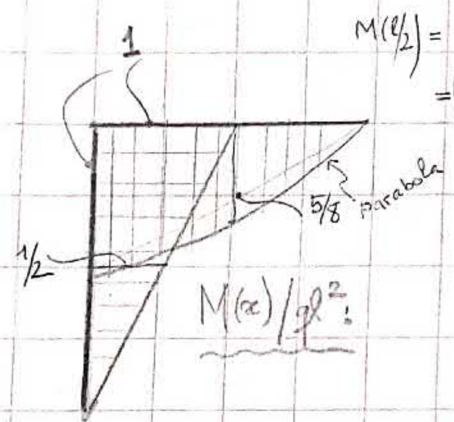
$$= \frac{ql^4}{1}\left[\frac{1}{3} + \frac{9}{6 \cdot 4}\right] = ql^4 \cdot \left(\frac{1 \cdot 8}{3 \cdot 8} + \frac{9}{24}\right)$$

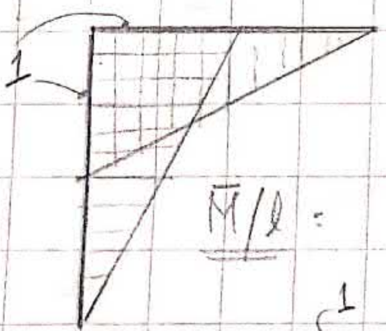
$\frac{8+9}{24} = \frac{17}{24}$

$$M(l/2) = \frac{3}{2} ql \cdot \frac{l}{2} + q\frac{l}{2} \cdot \frac{l}{4} = \left(\frac{3 \cdot 2}{4 \cdot 2} - \frac{1}{8}\right) ql^2 = \frac{5}{8} ql^2$$

$$EI u_B = \frac{17}{24} ql^4 \leftarrow \text{bending only}$$

$$u_C(\text{Bending}) = \frac{17}{24} ql^4 / EI$$

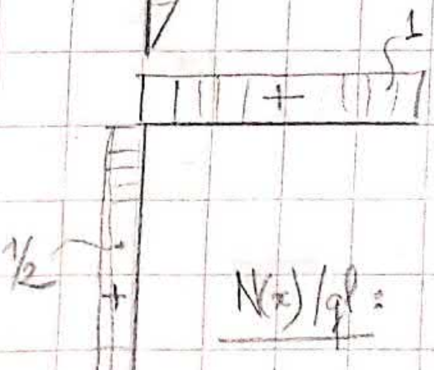


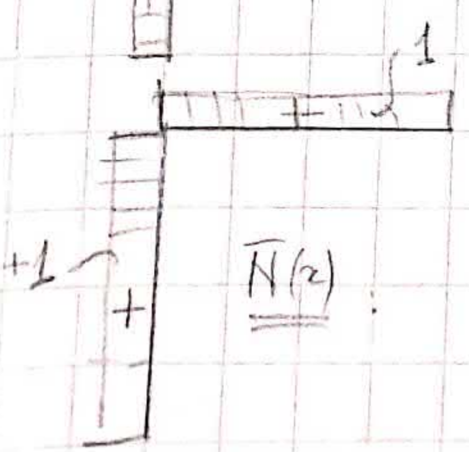


Axial force effects only: (Vereshagin integration rule)

$$EA u_N = \frac{1}{EA} \cdot \Big[\underbrace{ql^2 \cdot 1 + \frac{ql}{2} \cdot l}_{3/2\, ql^2}\Big] = \frac{3}{2} \frac{ql^2}{EA}$$

$$\boxed{U(C) = \frac{17}{24} \frac{ql^4}{EI} + \frac{3}{2} \frac{ql^2}{EA}}$$





Note: If you do not feel 'fluent'*) with this part yet, please spend a bit more time and train it until it becomes easy.

The future engineer in you deserves this knowledge.

*) If you need more than 2 pages to solve this

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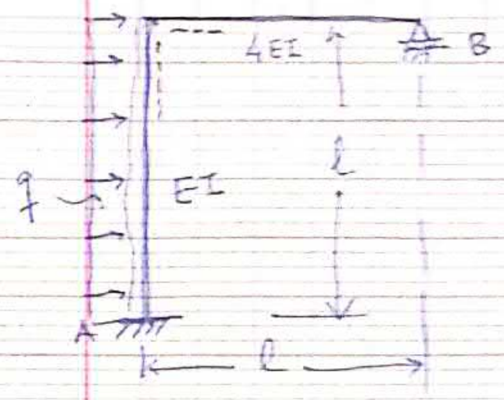
EX2

M(A) = ?

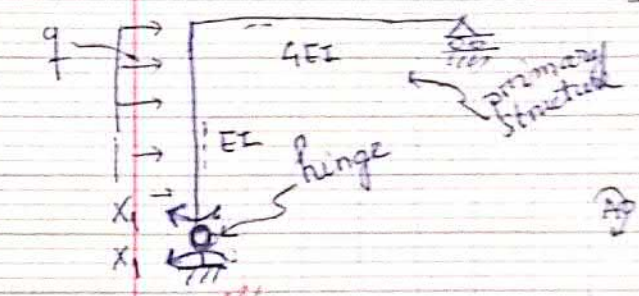


$n_s = 1$

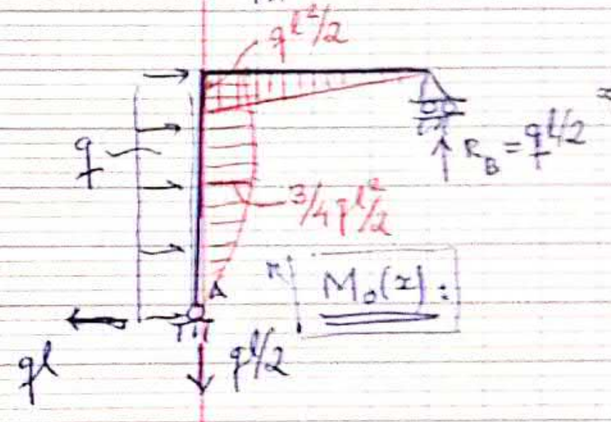
$\delta_{11} X_1 + \delta_{10} = 0 \rightarrow M(A) = M_0(A) + X_1 \cdot \overline{M}_1(A)$

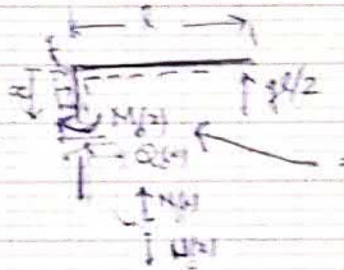
$\delta_{11} = \sum \int_s \overline{M}_1^2 / EI \cdot ds \rightarrow \overline{M}_1(s) = ?$

$\delta_{10} = \sum \int_s \overline{M}_1 M_0 / EI \cdot ds \rightarrow M_0(s) = ?$



$\sum M_A : q\ell \cdot \frac{\ell}{2} - R_B \cdot \ell = 0 \Rightarrow R_B = q\ell/2$





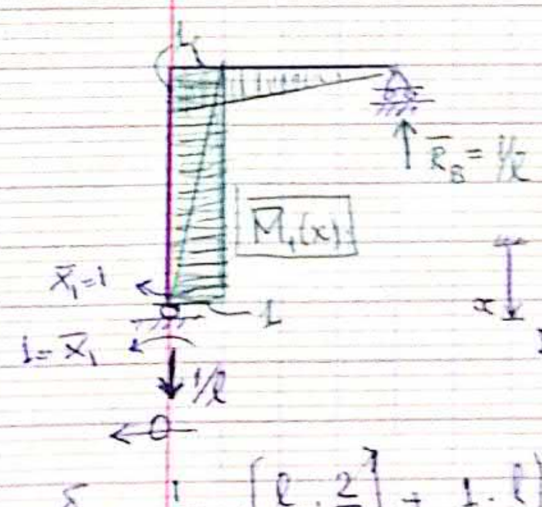
$\sum M_x : M(x) + q x^2/2 - q\ell^2/2 = 0$

$\Rightarrow M(x) = q\ell^2/2 - q x^2/2$

$= \frac{q\ell^2}{2}\left(1 - \left(\frac{x}{\ell}\right)^2\right)$

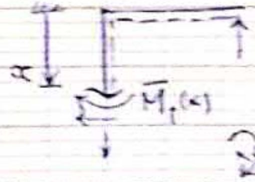
$M(\ell/2) = \frac{q\ell^2}{2} \cdot \underbrace{\left(1 - \frac{1}{4}\right)}_{=3/4} = 3/4\, q\ell^2/2$

$M(\ell) = 0$



$\sum M_A : \overline{R}_B \cdot \ell - 1 = 0 \Rightarrow \overline{R}_B = 1/\ell$

$\sum : 1/\ell \cdot \ell - 1 = 0$ ok



$\overline{M}_1(x) - 1/\ell \cdot \ell = 0 \Rightarrow \overline{M}_1(x) = 1$

$$\delta_{11} = \frac{1}{4EI}\left(\frac{\ell}{2}\cdot\frac{2}{3}\right) + \frac{1\cdot\ell}{EI} = \frac{\ell}{4E}\;\frac{\ell}{12EI} + \frac{\ell}{EI} = \frac{13}{12}\frac{\ell}{EI} \quad (\div 4EI)$$

$$\delta_{10} = \underbrace{\frac{1}{4EI}\cdot\frac{\ell}{2}\cdot\frac{2}{3}\,\frac{q\ell^2}{2}}_{\text{Verishagin}} + \underbrace{\frac{1}{EI}\cdot\frac{\ell}{6}\cdot\left[\frac{q\ell^2}{2}\underbrace{\left(1\cdot1 + 4\cdot\frac{3}{4}\cdot1 + 0\cdot1\right)}_{=4}\right]}_{\text{Simpson}}$$

$$= \frac{q\ell^3}{2\cdot EI\cdot 12} + \frac{q\ell^3}{EI\cdot 3} = \frac{q\ell^3}{EI}\cdot\left(\frac{1}{24} + \frac{1\cdot 8}{3\cdot 8}\right) = \frac{9}{24}\cdot\frac{q\ell^3}{EI}$$

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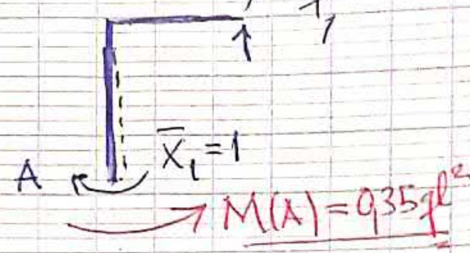
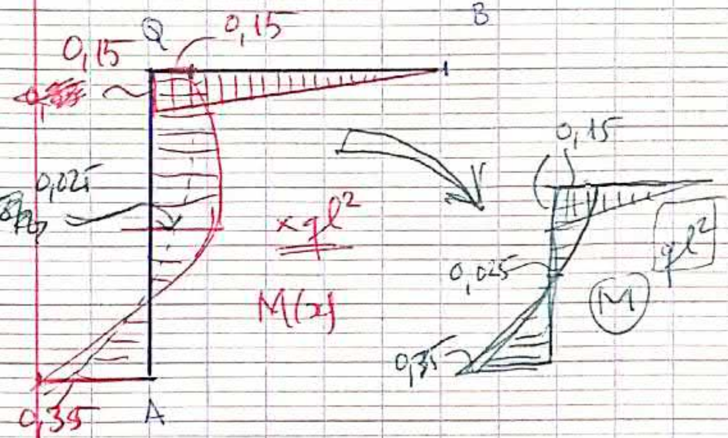
$$X_1 = -\frac{\delta_{10}}{\delta_{11}} = -\frac{9}{24}\frac{ql^3}{EI}\cdot\frac{12EI}{13l} = -\frac{9}{26}ql^2 \approx -\underline{0{,}35\,ql^2}$$

OK

$M(A) = -0{,}35\,ql^2$ means that the direction of $M(A)$ is opposite to the dir. of $\overline{X}_1$





Support reaction

$\frac{M(Q)}{l} = 0{,}15\,ql$

$$\underline{M = M_0 + X_1\cdot\overline{M}_1(x)} \longleftarrow \text{superposition}$$

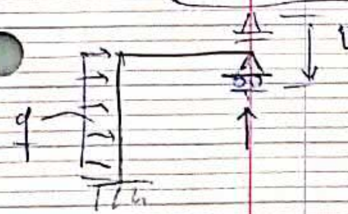
$$M(Q) = M_0(Q) + X_1\cdot\overline{M}_1(Q) = ql^2/2 - 0{,}35\,ql^2\cdot 1 = 0{,}15\,ql^2$$

$$M(x) = M_0(x) + X_1\underbrace{\overline{M}_1(x)}_{\equiv 1} = \frac{ql^2}{2}\left(1-\left(\frac{x}{l}\right)^2\right) - 0{,}35\,ql^2$$

$x = l/2 \sim \left(1-\frac{1}{4}\right) = 3/4 - 0{,}35$

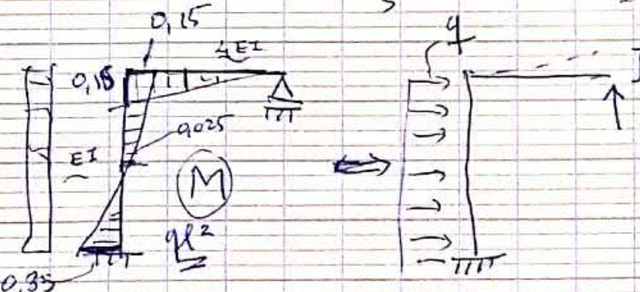
$x = l/2$

cross-check $v(B) = ? = 0$ ⟵ should be = to zero

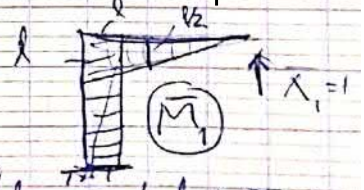


$$v(B) = \Sigma\int_s \overline{M}\kappa\,dx = \Sigma\int M(x)\cdot\frac{\overline{M}_1}{EI}\,ds$$

$v(B) = 0$, is it?



$R = \frac{0{,}15\,ql^2}{l} = 0{,}15\,ql$ =2/13 q*L



$$0 \overset{?}{=} v(B) = \Sigma\int \frac{M\overline{M}_1}{EI}\,dx = \frac{ql^2}{4EI}\left(l^2\cdot\frac{1}{2}\cdot 0{,}15\cdot\frac{2}{3}\right) + \frac{l\cdot ql^2}{6EI}\left(l\cdot 0{,}15 + 4\cdot l\cdot 0{,}025 + \ldots\right.$$

$$\left.-0{,}35\cdot l\right) = \frac{ql^4}{4EI}\cdot\left(0{,}15/3\right) + \frac{ql^4}{6EI}\cdot\left(0{,}15 + 4\cdot 0{,}025 - 0{,}35\right)$$

$$= \frac{ql^4}{EI}\cdot\left(\frac{0{,}15}{3\cdot 4} + \frac{0{,}15}{6} + \frac{4\cdot 0{,}025}{6} - 0{,}35/6\right) =$$

$$= \frac{ql^4}{EI}\left(\frac{0{,}075 + 0{,}15 + 4\times 0{,}025 - 0{,}35}{6}\right) = \frac{ql^4}{\ldots}\left(0{,}325 - 0{,}35\right) \approx 0 \text{ ok}$$

$0{,}15 \,/\, \frac{2}{7{,}5}$

0,075 + 0,15 = 0,225

0,025 × 4 = 0,100

+ 0,225

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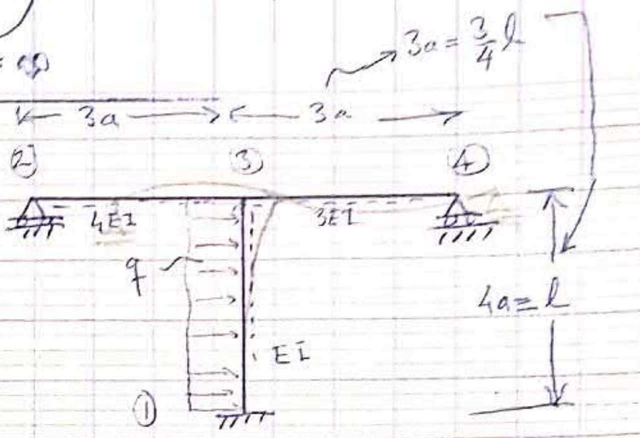
$15 \times 4 / 3 = (15 \times 4) \div 3$ $\quad \underbrace{15 \times 4}_{60} : 3 = \varphi$

5

$+ \; -$

Ex3 :

everything is here



$$\left.\begin{cases} \varphi_{32} = \varphi_{34} = \varphi_{31} \equiv \varphi_3 \\ \varphi_{13} = 0 \\ \cdot\ \psi_{13} = \psi_{31} \equiv \psi_3 \end{cases}\right] \begin{matrix} (1) \\ n_k = 2 \\ (2) \end{matrix} \quad \text{2 unknowns}$$

the problem is solved

two eqs. of equilibrium

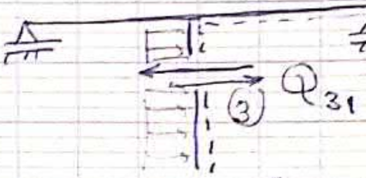
2 equilibrium eq.

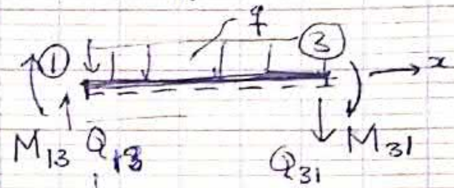
$$\boxed{\begin{cases} M^0_{32} + M^0_{34} + M_{31} = 0 & (1) \\ Q_{31} = 0 & \leftarrow (2) \end{cases}}$$

$l_{13} \equiv l$

$$\Downarrow$$

$$\cdot\ M_{13} + M_{31} + q\, l^2/2 = 0 \qquad (2')$$





$$0 = M_{13} + M_{31} + Q_{31}\, l_{31} + q\, l_{31}^2/2$$

$$-Q_{31} = +\frac{M_{13} + M_{31}}{l_{13}} + q\frac{l_{13}}{2}$$

$$(1): \begin{cases} M^0_{32} = a^0_{32}\varphi_3 \\ M^0_{34} = a^0_{34}\varphi_3 \\ M_{31} = a_{31}\underbrace{\varphi_{31}}_{\equiv \varphi_3} + b_{31}\cancel{\varphi_{13}}^{0} - c_{31}\overbrace{\psi_{31}}^{\equiv \psi_3} + \overline{M}_{31} \end{cases}$$

$$(2'): \; M_{13} = a_{13}\cancel{\varphi_{13}}^{0} + b_{13}\underbrace{\varphi_{31}}_{=\varphi_3} - c_{13}\overbrace{\psi_{13}}^{=\psi_3} + \overline{M}_{13}$$

$$\begin{bmatrix} a^0_{32} + a^0_{34} + a_{31} & -c_{31} \\ \underbrace{b_{13} + a_{31}}_{= c_{13}} & -2c_{13} \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} \overline{M}_{31} \\ \overline{M}_{13} + q\frac{l_{13}^2}{2} + \overline{M}_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\overline{M}_{13} + \overline{M}_{31} = 0$

$\overline{M}_{31} = \frac{q l^2}{12}$

$$\Downarrow$$

$$\begin{bmatrix} \overset{4 \cdot 3EI/l_{32}}{a^0_{32}} + \overset{3\cdot(3EI)/l_{34}}{a^0_{34}} + \overset{4EI/l_{31}}{a_{31}} & -c_{13} \\ -c_{13}^{\;6EI/l_{13}} & 2c_{13} \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \psi_3 \end{bmatrix} = + \begin{bmatrix} -\overline{M}_{31} \\ +q\, l_{13}^2/2 \end{bmatrix} = \frac{q\, l_{13}^3}{2EI}\begin{pmatrix} -1/6 \\ 1 \end{pmatrix}$$

symmetric stiffness matrix

$$\begin{bmatrix} 32 & -6 & -1/12 \\ -6 & 12 & +1/2 \end{bmatrix} \quad = \frac{q\, l_{13}^3}{EI} \cdot \begin{bmatrix} -\frac{1}{12} \\ \frac{1}{2} \end{bmatrix}$$

$12 + 13 = 25$

$$\underline{\underline{K}} = \frac{EI}{l} \cdot \begin{bmatrix} \underbrace{\overset{16}{3\cdot(4)} + \overset{+12}{3\cdot(3)} + \overset{+4}{4}}_{=25} & -6 \\ -6 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 32 & -6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} \varphi_3 \\ \psi_3 \end{bmatrix} = + \frac{q\, l^3}{12EI}\begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

(6.1) $a_{32}^{0} = 16EI/l$; $a_{34}^{0} = 12EI/l$; $a_{31} = 4EI/l$; $c_{13} = 6EI/l$

$$\frac{EI}{l}\cdot\begin{bmatrix}32 & -6\\ -6 & 12\end{bmatrix}\begin{bmatrix}\varphi_3\\ \psi_3\end{bmatrix} = -ql^2\begin{bmatrix}1/12\\ -1/2\end{bmatrix} \longrightarrow \begin{bmatrix}32 & -6\\ -6 & 12\end{bmatrix}\begin{bmatrix}\varphi_3\\ \psi_3\end{bmatrix} = \frac{ql^3}{EI}\begin{bmatrix}-1/12\\ +1/2\end{bmatrix}$$

$$\begin{bmatrix}\varphi_3\\ \psi_3\end{bmatrix} = +\,ql^3/EI\cdot\begin{bmatrix}0,0057\\ 0,0445\end{bmatrix}$$

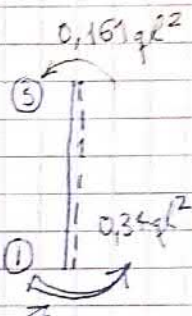
= q a^3/EI * {32/87, 248/87}

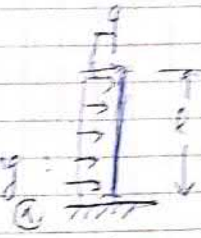
$u_3 = \psi_3\cdot l = 0,0445\, ql^4/EI$

= 992/87 * q*a^4/EI

$M_{13} = b_{13}\varphi_3 - c_{13}\psi_3 + \bar{M}_{13} = -0,34\, ql^2$

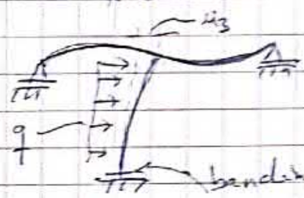
= -472/87 * q * a^2

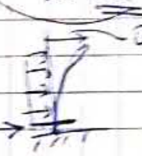




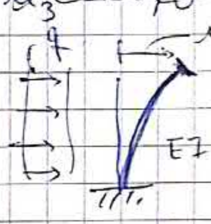
Checking: For a free cantilever $M_1 = ql^2/2$. So our result $0,34ql^2 < 0,5ql^2$ is ok

The frame





$\bar{u}_3 = 0,0$



$u = 0,125\, ql^4/EI$

checking

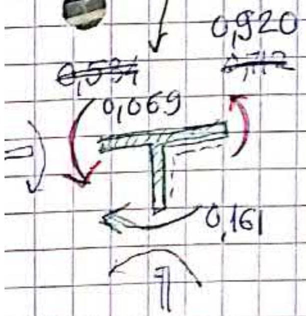
u_3 ok

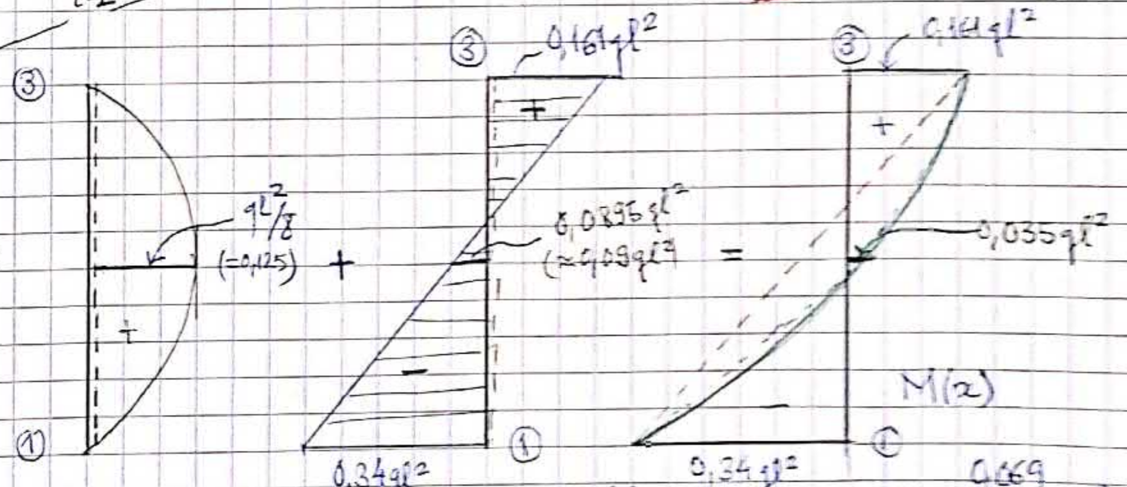
— o 0 o —

$$\bar{M}_{31} = a_{31}\varphi_3 + b_{31}\cancel{\varphi_{13}} - c_{13}\psi_3 + \bar{M}_{31} = \frac{4EI}{l}\cdot 0,0057\frac{ql^3}{EI} - \frac{6EI}{l}\cdot 0,0445\frac{ql^3}{EI} + \frac{ql^2}{12}$$

$$= -0,161\, ql^2$$

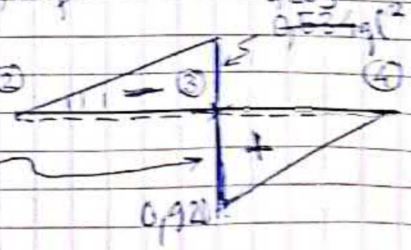
**) verification: do we have $\sum_{i=2}^{4} M_{3i} = 0$? Yes





$$M_{34}^{0} = a_{34}^{0}\varphi_3 = \frac{12EI}{l}\,\overset{0,057}{\cancel{0,0445}}\,\frac{ql^3}{EI} = \overset{0,069\,ql^2}{\cancel{-0,534\,ql^2}}$$

$$M_{32}^{0} = a_{32}^{0}\varphi_3 = \frac{16EI}{l}\cdot\overset{0,057}{\cancel{0,0445}}\,\frac{ql^3}{EI} = \underset{0,920}{\cancel{0,712}}\,ql^2$$



**) This verification allowed me to detect a mistake.
So, get use to do cross-checks. No one is free of errors.

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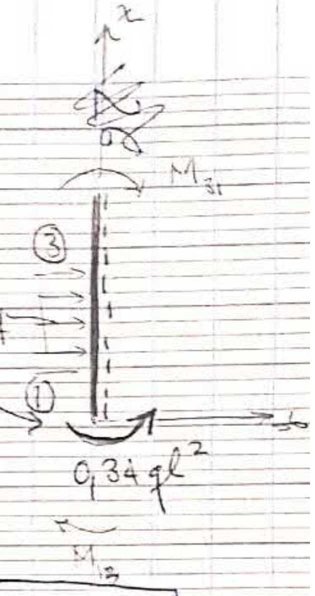
$$\underline{M_{13}} = b_{13}\varphi_3 - a_{13}\psi_3 + \overline{M}_{13}$$

$$= \frac{2EI}{l}\varphi_3 - \frac{6EI}{l}\psi_3 + \left(-ql^2/12\right)$$

$\varphi_3 = 0{,}0057\, ql^3/EI$ $\psi_3 = 0{,}0455 \cdot ql^3/EI$

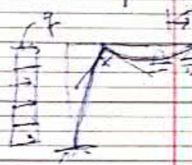
$$= \underbrace{\left(2 \cdot 0{,}0057 - 6 \cdot 0{,}0455 - 1/12\right)}_{*\, ql^2} = \underline{-0{,}34\, ql^2}$$



$$\underline{M_1 = -0{,}34\, ql^2}$$

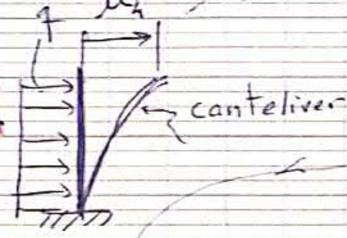
$$\rightarrow u_4 = \psi_{13} \cdot l = 0{,}0455\, \frac{ql^3}{EI} \cdot l$$

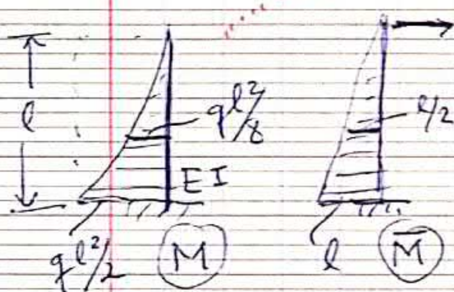
Horizontal displacement



$$= \underline{0{,}0455\, \frac{ql^4}{EI}}$$

cross-checking: $u_4 < \tilde{u}_4$





$$\tilde{u}_4 = \int_0^l \frac{M \cdot \overline{M}}{EI}\, ds = \frac{l}{6EI}\left[0 \cdot 0 + 4 \cdot \underbrace{\frac{ql^2}{8} \cdot \frac{l}{2}}_{1/4} + \underbrace{\frac{ql^2}{2} \cdot l}_{1/2}\right]$$

$$= \frac{3}{26 \cdot 4}\, q\frac{l^4}{EI} = \underline{\frac{1}{8}\, q\frac{l^4}{EI}}$$

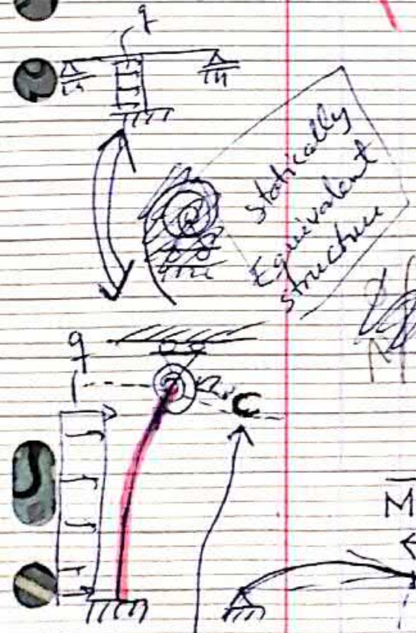
$1/4 + 1/2 = 3/4$

$\frac{1}{2} \cdot \frac{1}{4} \rightarrow 0{,}25 : 2 = \underline{0{,}125}$

$$u_4 = 0{,}046\, \frac{ql^4}{EI} < \tilde{u}_4 = \underline{0{,}125\, ql^4/EI}$$

ok

I let the student find out why (to show) u_4 is less than $\tilde{u}_4$

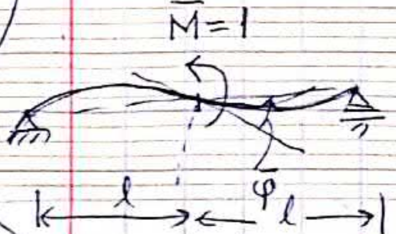


$$U_{\text{frame}}^{F} = U_{\text{column}}^{F} + U_{\text{beam}}^{F} > U_{\text{cantilever}}^{F}$$

$\underbrace{U_{\text{column}}}_{\geq 0}$ $\underbrace{U_{\text{beam}}}_{\geq 0}$

$$u_4 = \frac{\partial U_{\text{frame}}}{\partial F}\Big|_{(F \to 0)} > \frac{\partial U_{\text{cantilever}}}{\partial F}\Big|_{(F \to 0)} = \tilde{u}_4$$

Castigliano



$$\overline{M} = c\,\overline{\varphi}$$

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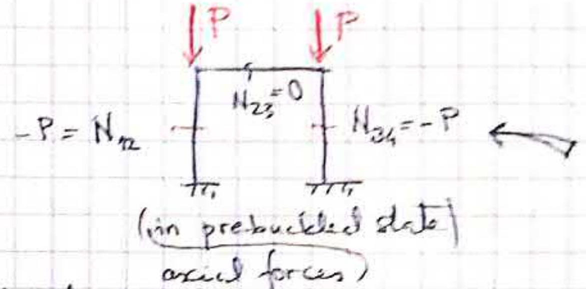
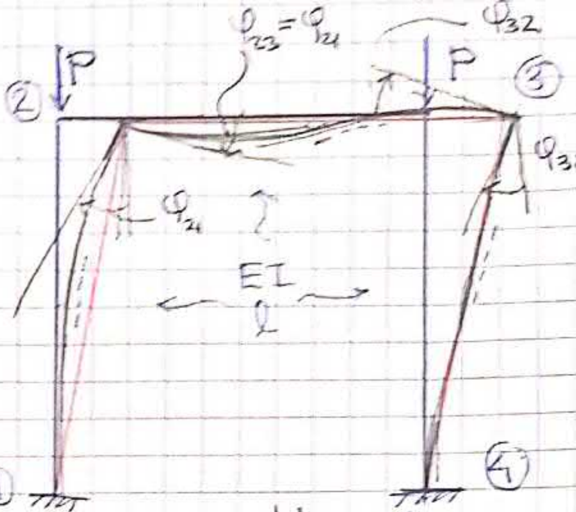
7 (4)





The possible buckling modes are in number of two: symmetric and antisymmetric. In principle both ~~the~~ associated buckling loads should be determined. The smallest of the two will be the buckling load and the associated buckling mode will be the buckling mode.

In the examination question we limit the answer to the antisymmetric case because of lack of time.

ANTISYMMETRIC BUCKLING (this will give the smallest buckling load).

$$(1)\ \varphi_{21}=\varphi_{23}=\varphi_{32}=\varphi_{34}\equiv\varphi \qquad \varphi_{12}=\varphi_{43}=0$$

$$(2)\ \psi_{21}=\psi_{34}\equiv\psi \qquad \psi_{23}=0$$

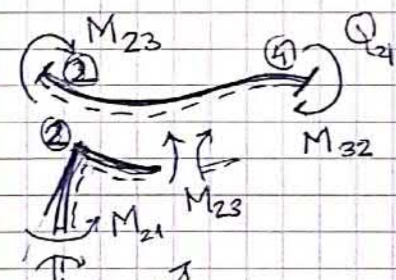
$m_\ell = 2$

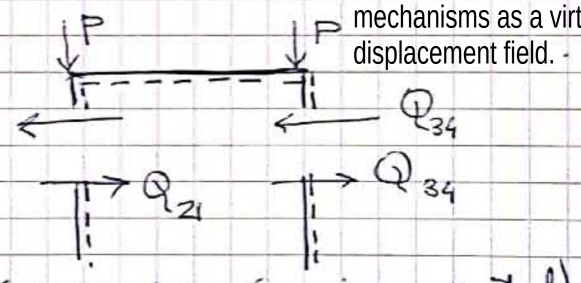
NEED 2 EQUILIBRIUM EQS.

We can also use the virtual work principle to derive the second equilibrium equation by taking the rigid-body motion of the sway mechanisms as a virtual displacement field.

$$(1')\ M_{21}+M_{23}=0$$

$$(2')\ Q_{21}+Q_{34}=0$$





(only shear force is represented)

First equation:

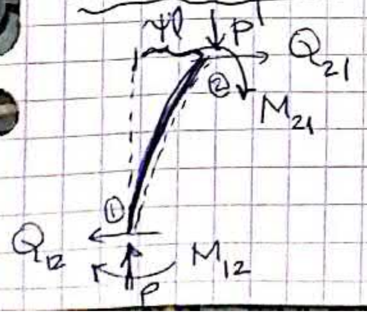
$$M_{21}=A_{21}(P)\underbrace{\varphi_{21}}_{\equiv\varphi}+B_{21}(P)\cancel{\varphi_{12}}^{0}-C_{21}\underbrace{\psi_{21}}_{\equiv\psi}=\underline{A_{21}(P)\varphi-C_{21}(P)\psi}$$

$$M_{23}=a_{23}\underbrace{\varphi_{23}}_{\equiv\varphi}+b_{23}\underbrace{\varphi_{32}}_{\equiv\varphi}-c_{23}\cancel{\psi_{23}}^{0}=\underbrace{(a_{23}+b_{23})}_{=c_{23}}\varphi_2=\underline{c_{23}\varphi}$$

$$(1')\Rightarrow A_{21}\cdot\varphi-C_{21}\psi+c_{23}\varphi=0\Rightarrow\boxed{[A_{21}(P)+c_{23}]\varphi-C_{21}(P)\cdot\psi=0}$$

Second equation:

$$(2)\ \boxed{P\cdot\psi\ell+M_{21}+Q_{21}+M_{12}=0}\ (3)$$



$$\text{from } (2')\ Q_{21}+\underbrace{Q_{34}}_{=Q_{21}\ \leftarrow\text{by symmetry}}=0\Rightarrow 2Q_{21}=0\Rightarrow \underline{Q_{21}=0}$$

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$\boxed{8}$

(3) and $Q_{21} = 0$ give: $\boxed{P \cdot \psi \cdot l + M_{21} + M_{12} = 0}$

$$M_{21} = A_{21}(P)\varphi - C'_{21}(P)\cdot\psi$$

$$M_{12} = A_{12}\cancel{\varphi_{12}} + B_{12}(P)\varphi - C_{12}(P)\psi$$

$$\Rightarrow \boxed{P\cdot\psi\cdot l + \underbrace{[A_{21}(P) + B_{12}(P)]}_{=C_{21}(P)}\varphi - \underbrace{(C_{12}(P) + C_{21}(P))}_{2C_{12}(P)}\psi = 0} \quad (2'')$$

Now, let's arrange eqs. (1') and (3) in MATRIX-form.

$$\underbrace{\left[\begin{array}{c|c} A_{21}(P) + \overset{C_{23}(P=0)}{C_{23}} & -C_{21}(P) \\ \hline C_{21}(P) & -2C_{12}(P) + P\cdot l \end{array}\right]}_{\underline{\underline{K}}} \underbrace{\begin{bmatrix}\varphi \\ \psi\end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{0}} \rightarrow \underline{\underline{K}}\,\underline{q} = \underline{0}$$

$$\det\left[\underline{\underline{K}}(P)\right] = 0 \rightarrow P_{cr} = \dots \quad \text{smallest root}$$

Notation:

$$kl \equiv \lambda = \sqrt{\frac{P}{EI}}\cdot l \rightarrow \lambda^2 = \frac{P}{EI}\cdot l^2$$

$$P = \frac{\lambda^2 EI}{l^2}$$

Just program the stiffness matrix as a function of λ. Then plot $\det(\underline{\underline{K}}(\lambda))$. Find the smallest root.

$$\underline{\underline{K}} = \left[\begin{array}{c|c} A_{21}(\lambda) + C_{23} & -C_{21}(\lambda) \\ \hline C_{21}(\lambda) & \lambda^2 \cdot EI/l - 2C_{12}(\lambda) \end{array}\right]$$

$$\det(\underline{\underline{K}}) = (A_{21}(\lambda) + C_{23})\left(\lambda^2 EI/l - 2C_{12}(\lambda)\right) - C_{21}^2(\lambda) = 0$$

$$\begin{cases} A_{21} = \dfrac{4EI}{l}\cdot\dfrac{3\psi(\lambda)}{D(\lambda)} \\[2ex] B_{21} = \dfrac{2EI}{l}\cdot\dfrac{3\phi(\lambda)}{D(\lambda)} \end{cases} \qquad C_{21} = A_{21} + B_{21} = \frac{EI}{l\,D(\lambda)}\cdot 6(\psi(\lambda) + \phi(\lambda))$$

$$\underline{\underline{K}} = \left[\begin{array}{c|c} \dfrac{4EI}{l}\cdot\dfrac{3\psi(\lambda)}{D(\lambda)} + \dfrac{6EI}{l} & -\left(\dfrac{4EI}{l}\cdot\dfrac{3\psi(\lambda)}{D(\lambda)} + \dfrac{2EI}{l}\dfrac{3\phi(\lambda)}{D(\lambda)}\right) \\ \hline \dfrac{EI}{l}\dfrac{1}{D(\lambda)}\cdot 6(\psi(\lambda) + \phi(\lambda)) & \lambda^2\cdot\dfrac{EI}{l} - 2\cdot\dfrac{EI}{l}\dfrac{1}{D(\lambda)}\cdot 6(\psi(\lambda) + \phi(\lambda)) \end{array}\right] =$$

$$\underline{\underline{K}} = \frac{EI}{l\,D(\lambda)}\cdot\left[\begin{array}{c|c} 12\psi(\lambda) + 6 & -6(\psi(\lambda) + \phi(\lambda)) \\ 6(\psi(\lambda) + \phi(\lambda)) & \lambda^2\cdot D(\lambda) - 12(\psi(\lambda) + \phi(\lambda)) \end{array}\right]$$

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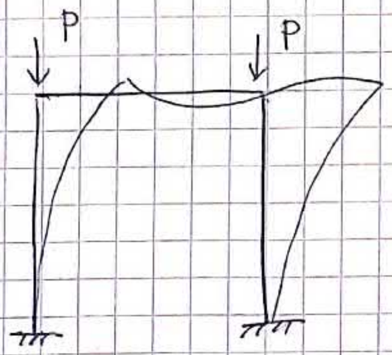
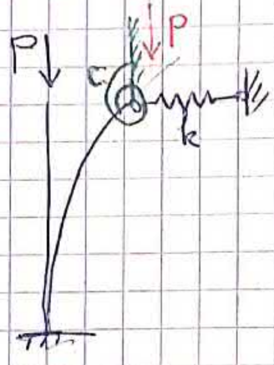
9



$\Longleftrightarrow$

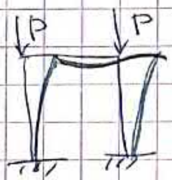


P_{cr} $<$ (much smaller)

P_{cr} $>$

P_{cr} $<$

$<$ P_{cr}



$<$

$\underbrace{P_E = 4 \cdot \pi^2 \frac{EI}{\ell^2}}_{\text{too stiff}}$

$P_E = 0{,}25\pi^2 \frac{EI}{\ell^2}$ (too loose) $<$ $P_{cr} = \lambda_{cr}^2 EI/\ell^2$ $<$

$\simeq 0{,}75\,\pi^2 EI/\ell^2$

serve as good bracket to be refined by $\det(\underline{\underline{K}}(\lambda)) = 0$.

$\det(\underline{\underline{K}}(\lambda)) = 0 \Rightarrow \lambda_{min} = 2{,}7164597\ldots$ $\lambda^2_{min} =$

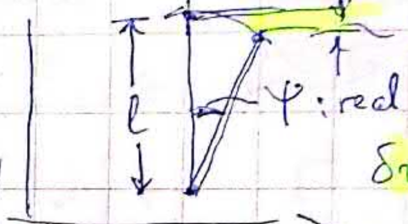
$\lambda^2_{min}/\pi^2 \approx 0{,}748 \rightarrow P_{cr} \approx 0{,}748\,\pi^2 EI/\ell^2 \in [0{,}25 \div 4]\pi^2$ ok

$\lambda^2 = (k\ell)^2 = \ell^2 \frac{P}{EI} \rightarrow P = \lambda^2 \cdot \frac{EI}{\ell^2} = \left[\frac{\lambda^2}{\pi^2}\right] \cdot \pi^2 \frac{EI}{\ell^2}$

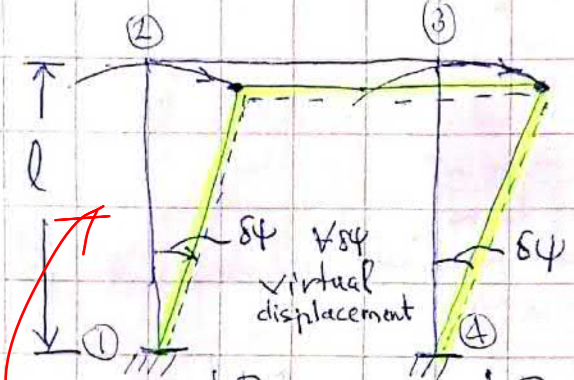
ved graphically

Ex. 4

Virtual work version for stability (shear equation)



$$\delta v = +\delta\left(l(1-\cos\psi)\right)$$

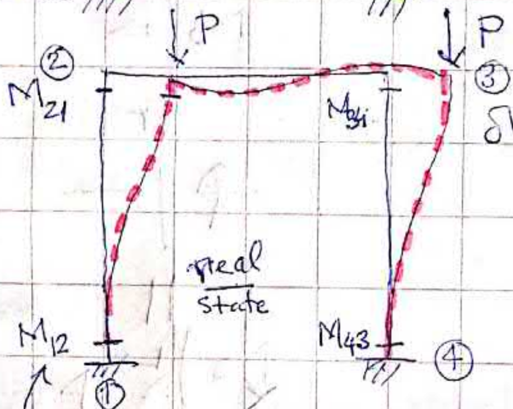


$$\delta v = l\sin\psi\cdot\delta\psi \approx l\psi\,\delta\psi$$

for moderate rotations

$$= \delta(-l\cos\psi) = -l\,\delta(\cos\psi) = +l\sin\psi\,\delta\psi$$

the idea is to account for the higher order displacements (= geometrical nonlinearity)



$$\delta W_{ext} + \delta W_{int} = M_{12}\delta\psi + M_{21}\delta\psi + M_{43}\delta\psi + M_{31}\delta\psi + 2P\cdot\delta v = 0$$

because of (anti) symmetry in the buckling mode

$$\delta W = (M_{12} + M_{21})\cdot 2\delta\psi + 2P\,l\,\psi\,\delta\psi = 0,\ \forall\delta\psi$$

$$\Downarrow \quad \boxed{M_{21} + M_{12} + Pl\psi = 0} \quad (4)$$

end-moments

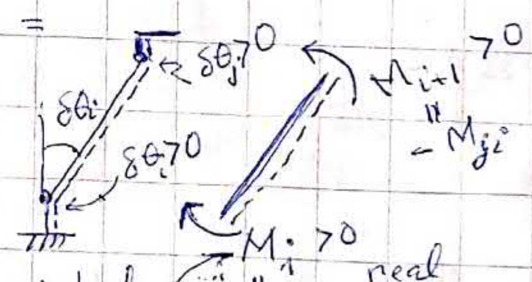
this equilibrium equation is the same as the shear (equilibrium equation) (3) derived previously on page 7

---

definition: $\delta W_{int} = -\sum M_i\delta\theta_i = -\sum_i \underbrace{M_i}_{\text{bending moment}}\cdot\underbrace{\delta\theta_i}_{\text{virtual rotation}} =$

cf. $\delta W_{int} = -\int_l M\,\underbrace{\delta\kappa\,dx}_{\delta\theta}$

We can also use the virtual work principle to derive the second equilibrium equation by taking the rigid-body motion of the sway mechanisms as a virtual displacement field.



$$= -M_{ij}(-\delta\theta_i) - (-M_{ji})\delta\theta_j$$

$$= M_{ij}\,\delta\theta_i + M_{ji}\cdot\delta\theta_j$$

this is why

$$\delta W_{int} = +\sum M_{ij}\,\delta\theta_{ij}$$

plus, now sign

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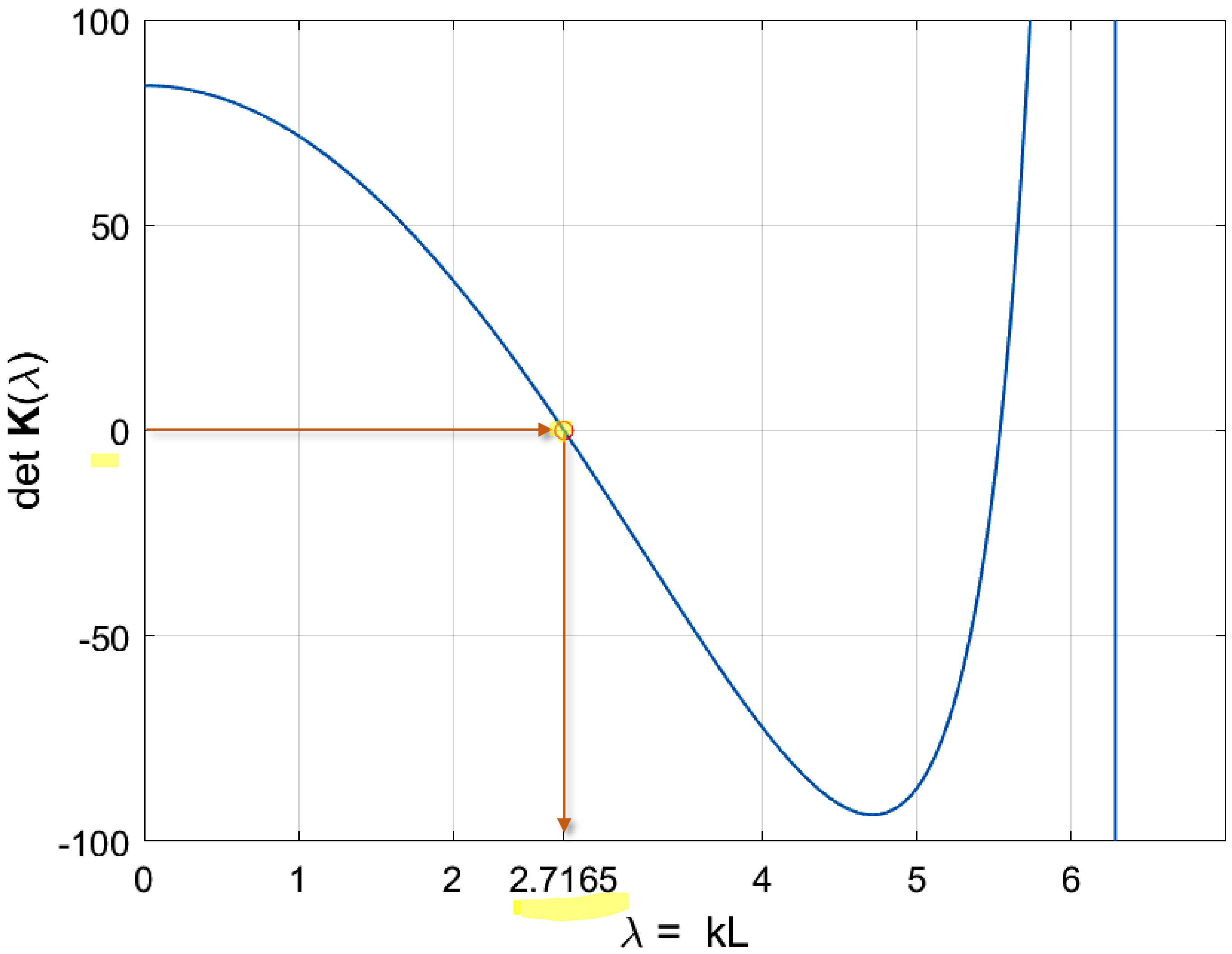
Solving graphically for the smallest eigenvalue = critical buckling load

```
% exam   26.10.2021
%-----------------
clear
clc

EI_per_L = 1;     % EI/L is factorized  out of the determinant
                  %  EI/L * det (K)  = 0;
                  %  we compute det(K)
                  % ------------------------------------------

lam    = 0.001:1/1000:8;
determ = zeros(size(lam));

for i = 1:length(lam)

[FI, PSI] = Berry(lam(i));   %---- returns Berry's functions
D = 4 * PSI .* PSI  - FI .* FI;

A_12 = 2*6 * PSI ./ D;
B_12 =    6 * FI ./ D;
C_12 = A_12 + B_12;

c_23 = 6;
K_11 = A_12 + c_23;
K_12 = - C_12;
K_21 = K_12;
K_22 = 2* C_12 - lam(i) .* lam(i);

K = [K_11   K_12
     K_21   K_22];

determ(i) = det(K) ;
end
  figure
  plot(lam, determ)
  grid on
  axis([0 9 -100 100])
```

```
function [FI, PSI] = Berry(lam)
% yhdistetty puristetus  ja taivutetus
% combined axial compression  and bending
  FI  = 6.0 * ( - 1 ./ lam + 1 ./ sin(lam) ) ./ lam;
  PSI = 3.0 * ( + 1 ./ lam - 1 ./ tan(lam) ) ./ lam;
return
```



$$\det(\underline{\underline{K}}) = \left(A_{21}(\lambda) + C_{23}\right)\left(\lambda^2 EI/\ell - 2C_{12}(\lambda)\right) - C_{21}^2(\lambda) = 0$$

```
% EI/L is factorized  out of the determinant
%  EI/L * det (K)  = 0;
%  we compute det(K)
% ------------------------------------------
```

is obtained. Finally, equation (148) can be written in a no-dimensional form for the *long-waited-for* interaction diagram

$$\left(\frac{\omega_n}{\bar{\omega}}\right)^2 = 1 - \frac{P}{P_{cr,n}}. \qquad (151)$$

The graph (Fig. **??**). of this relation cannot be simpler: a straight line going from value 1 to value 1. Note that the graph may be curved for other boundary conditions.

START HERE

3 Stability analysis of a sway frame

In the following both static and dynamic approach will be shown with an application example of a sway[30] simple frame (Fig. 9).

Civil and mechanical engineers use the term of *stability loss* for a transition of the structure from one equilibrium state to another under the same loading. The new equilibrium configuration can be relatively close or far from the initial one. In the case of close neighbouring equilibrium, it is often question of bifurcational loss of stability (when many solutions appear for the same loading), like for example, column buckling. When the new equilibrium configuration is relatively far, it is often question of 'snap-through'-like loss of stability of a shallow arch, for instance. In this case, the transition occurs through a limit point. The number of solutions remains one.

Anyway, what I wanted to tell is that loss of equilibrium is by nature a dynamic phenomena where the 'excess' of input energy is dissipated as kinetic energy. For conservative systems some crucial aspects of the stability loss, like finding buckling load and modes, can be treated as static stability loss[31].

3.1 Static stability

Here the classical displacement method is used in its simplest form that civil engineers call *slope deflection method*[32]. The geometric non-linearity is introduced while solving deflections for combined axial forces and bending. This way, the obtained stiffness matrix depends non-linearly on the amount of compression (or tension).

In this example, there is two lower 1st and 2nd buckling modes: anti-symmetric and symmetric one. The anti-symmetric will give the lowest buckling load. For shortness, we consider here only the anti-symmetric mode. The symmetric mode is left for the student to train and show that, effectively, it

[30] Sivusiirtyvä kehä tai myös sivullesiirtyvä kehä (FI)

[31] There is a good concise course on stability of structures in the department of civil engineering in Otaniemi (Aalto university).

[32] Kulmanmuutosmenetelmä (FI)

gives a higher buckling load. The reader have not to believe me. Check! However, I do not tell lies.

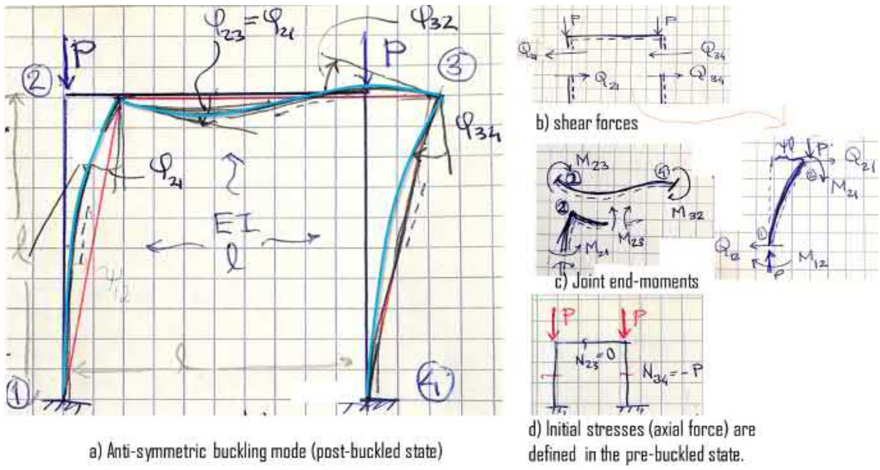


Figure 9: Buckling of a sway frame into anti-symmetric mode.

3.1.1 Buckling as an eigenvalue problem

Recall the reader that equilibrium equations are written in the slightly post-buckled configuration. The axial stresses (or more exactly, the initial stresses) are determined in the pre-buckled configuration close to buckling (Fig. 9d) as a consequence of the 'linearisation' of the geometrically non-linear equations of equilibrium at the buckling load (= bifurcation point) keeping up to quadratic terms in Taylor series.

Kinematics and equations of equilibrium: Assuming the anti-symmetric buckling mode (Fig. 9a) we have two kinematic independent unknowns, namely, ϕ and ψ defined as

$$\phi_{21} = \phi_{23} = \phi_{32} = \phi_{34} \equiv \phi \tag{152}$$

$$\psi_{21} = \psi_{34} \equiv \psi \tag{153}$$

where the remaining end-rotations (geometric boundary conditions)

$$\phi_{12} = \phi_{43} = 0 \tag{154}$$

and rigid-body rotations

$$\psi_{23} = 0. \tag{155}$$

Since there is two independent kinematic degrees of freedom (dofs), we need two independent equilibrium equations in which these dofs appear (Fig. 9b and c).

Consequently, the sufficient equilibrium equations are

$$
\begin{aligned}
M_{21} + M_{23} =& 0 && (156)\\
Q_{21} + \underbrace{Q_{34}}_{=Q_{21},\,\text{symmetry}} =& 0 \implies Q_{21} = 0 && (157)
\end{aligned}
$$

Note that, by symmetry, we have $Q_{34} = Q_{21}$. In addition, it is the above equilibrium equations (Eqs. 156) and 156) that will result in the eigenvalue problem

$$\mathbf{K}(\lambda) \cdot \mathbf{v} = \mathbf{0}, \text{ where } \mathbf{v} = [\phi,\, \psi]^{\mathrm{T}} \tag{158}$$

which solutions $(\lambda, \mathbf{v}_{(2\times1)})$ provide the buckling load λ_{cr} and the corresponding mode $\mathbf{v}$. The parameter λ is the loading parameter, in general. In particular, here, $\lambda = \ell\sqrt{P/EI}$. So, everything is said. What follow are technical steps to form the stiffness matrix $\mathbf{K}_{(2\times2)}$.

Pre-stress: The stiffness matrix depend on the axial force for a beam-element under combined compression (tension) and bending. The end-moments and end-rotations are related by stiffness relations. These stiffness-coefficients A_{ij}, B_{ij} and C_{ij} depend non-linearly on the axial force. The axial stress (or forces) in the bars are to be defined in the pre-buckled configuration prior buckling (Fig. 9d).

So,

$$N_{21} = N_{34} = -P, \text{ and } N_{23} = 0 \tag{159}$$

The needed end-moments are

$$
\begin{aligned}
M_{21} =& A_{21}(N_{21})\phi_{21} + B_{21}(N_{21})\underbrace{\phi_{12}}_{=0} - C_{21}(N_{21})\psi_{21} && (160)\\
=& A_{21}(N_{21})\phi - C_{21}(N_{21}) && (161)\\
M_{23} =& a_{23}\phi_{23} + b_{23}\underbrace{\phi_{32}}_{=\phi_{23}} - c_{23}\underbrace{\psi_{23}}_{=0} = (a_{23} + b_{23})\phi = c_{23}\phi && (162)\\
M_{12} =& B_{12}(N_{12})\phi - C_{12}(N_{12})\psi && (163)
\end{aligned}
$$

Finally, moment equilibrium equation (Eq. 156) results in

$$\underbrace{[A_{21}(P) + c_{23}]\phi - C_{21}(P)\psi = 0}_{\text{The 1st equilibrium equation}} \tag{164}$$

Expressing equilibrium equation (Eq. 157)[33] in terms of end-moments (9b) we obtain:

$$P \cdot \psi\ell + M_{21} + M_{12} + Q_{21} = 0 \implies -Q_{21} = \underbrace{P \cdot \psi\ell + M_{21} + M_{12} = 0}_{\text{The 2nd equilibrium equation}} \tag{165}$$

[33] This equation is known as the shear equation.

Now you need to express the end-moments as function of end-displacements. Then writing the two equilibrium equations into matrix form we obtain, finally,

$$\underbrace{\begin{bmatrix} A_{21}(\lambda)+c_{23} & -C_{21}(\lambda) \\ -C_{21}(\lambda) & -\lambda^2 EI/\ell + 2C_{12}(\lambda) \end{bmatrix}}_{=\mathbf{K}(\lambda)} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{166}$$

The above equation is the *eigenvalue problem*[34] that provides the buckling load as the smallest eigenvalue and the corresponding eigenvectors as the buckling mode. Note that in the above stiffness matrix, the term EI/ℓ can be factorised out allowing to find numerically eigenvalues as function of the load parameter λ or to solve directly (graphically) for the smallest root.

Now buckling occurs when we have a non-trivial solution. Therefore, the condition

$$\det\{\mathbf{K}(\lambda)\} = 0 \tag{167}$$

is known as criticality condition, and provides the critical buckling load as

$$\lambda_{\text{cr}} = 2.716 \implies P_{\text{cr}} = 0.748\pi^2 EI/\ell^2. \tag{168}$$

The first root of the criticality (non-linear) condition was determined graphically (Fig. 10).

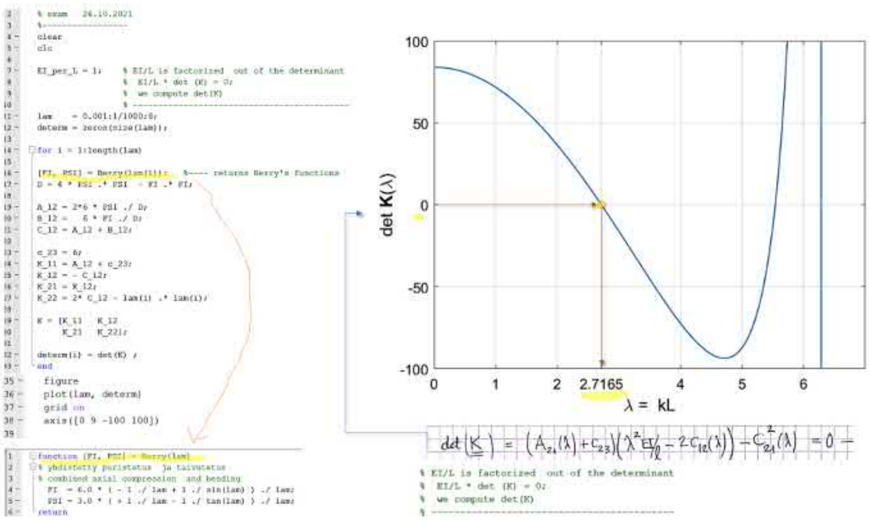


Figure 10: Graphical root solving.

[34]Ominaisarvotehtävä (FI). Pienin ominaisarvo antaa nurjahduskuorman ja vastaava ominaisvektori antaa nurjahdusmuodon.

3.1.2 The shear equation via virtual work principle

This is the story part: I am actually grading examination in structural mechanics. One of the exercises was the buckling of the frame above. I was very pleased when I found that four students from 77 derived the shear equation using the virtual work principle. Not only I was personally pleased but also for the students themselves who learned not only by heart the principle of virtual work but now they master using it for new, for them, situations. They have really an additional powerful tool at hand. This is why, I present here, also the shear equation derived using this principle for the 77-4 students. A gain, I emphasis that *no free-body diagram* is needed for that while for deriving the shear (equilibrium) equation, we need to find the correct free-body diagram with, sometimes, an explosion (= too much) of the internal forces at cuts to account for.

The virtual work principle is a poetry of mechanics that, in this particular example, allows to derive the shear equation without making cuts. It is almost magic.[35] We take the slightly post-buckled configuration as the real state and the sway-mechanism as the virtual state (Fig. 11). and write

$$\delta W_{\text{int}} + \delta W_{\text{ext}} = 0, \ \forall \delta\psi \tag{169}$$

The vertical displacement v under the load P is

$$v = \ell(1 - \cos\psi) \tag{170}$$

in which geometric non-linearity is embedded. Non-linearity allow for existence of multiple equilibrium configurations for the same loading. The resulting in a virtual displacement on which the load P will work is

$$\delta v = \ell \sin\psi \delta\psi \approx \ell\psi\delta\psi \tag{171}$$

where the last term is a second order Taylor expansion term (for moderate rotations) needed to obtain an linearised eigenvalue problem.[36] Now we can

[35]Think of the Schrödinger wave-equation of quantum physics (= mechanics). The primary mathematical objects it operates with (wave functions) are complex vectors in Hilbert spaces. However, the predictions that this quasi-mystical and mythical wave-equation provides are real physical objects and processes that are *observable*. For general knowledge, the need for Hilbert-spaces in quantum mechanics is not for beauty nor for luxury, or sometimes used for condensing writing. It is a necessity since the theory will not work if the vector spaces are not complex Hilbert spaces. Incredible! The virtual work principle sounds almost as mystical. However, the motion (or equilibrium) equations, that results are real and deal with physical objects and quantities (coordinates, velocities, . . .) that are measurable.

[36]If you feel that there are many technical subtle details then join the course I give on stability of structures in spring. It is a condensate of knowledge necessary that helps doing more safely structural analysis and design. In this course, we start from the beginning and everything becomes clear or at least clearer then before joining the course. The topics: Flexural buckling of beams, frames and continuous column-beams, plate buckling, pure torsional buckling, combined flexural-torsional buckling, lateral torsional buckling, buckling of cylindrical shells. Taivutusnurjahdus, levyjen ja laatojen lommahdus, vääntönurjahdus, avaruusnurjahdus, kiepahdus, pyörähdys-symmetristen kuorien lommahdus (FI).

write using end-moments[37]

$$\delta W_{\text{int}} + \delta W_{\text{ext}} = \underbrace{M_{12}\delta\psi + M_{21}\delta\psi + M_{34}\delta\psi + M_{43}\delta\psi}_{\delta W_{\text{int}}} + \underbrace{2P\delta v}_{\delta W_{\text{ext}}} \tag{172}$$

$$= \underbrace{2(M_{12} + M_{21})}_{\text{because of symmetry}} \cdot\delta\psi + 2P\delta v \tag{173}$$

$$=2(M_{12} + M_{21}) \cdot \delta\psi + 2P\psi\ell\delta\psi = 0,\ \forall\delta\psi \tag{174}$$

$$\implies \underbrace{(M_{12} + M_{21}) + P\psi\ell = 0}_{\text{(Eq. 157)}} \tag{175}$$

And we finally, recover the shear equation of equilibrium (Eq. 157) derived previously using a physical cut for the free-body diagram without using any free-body diagrams.

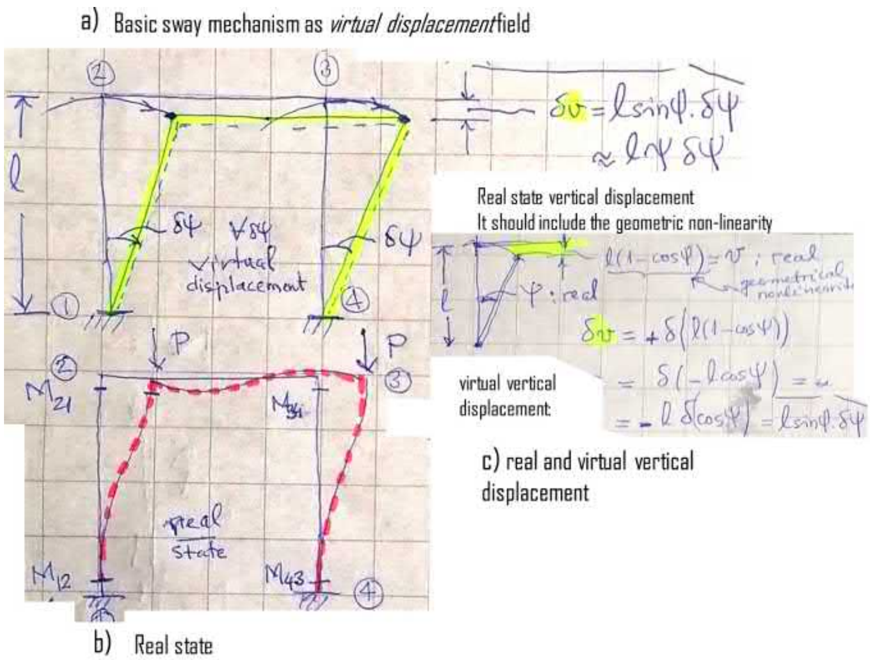


Figure 11: Virtual work principle used to derive equilibrium equations.

Must be noted that the virtual internal work (bending contribution) is primarily expressed for rigid body motion where relative rotations θ_i (or ψ_i) occurs at discrete locations i is given using the bending moments M_i by

$$\delta W_{\text{int}} = -\sum M_i\delta\theta_i \tag{176}$$

[37]Usually, virtual work of internal forces, in bending, is written using the bending moments. note that the definitions of bending and end-moments differ by a sign: $M_1 = M_{12}$ and $M_2 = -M_{21}$.

Now for a beam with ends (or nodes) 1 and 2, the relation between bending moments M_1, M_2 and end-moments M_{12} and M_{21} are defined such that

$$M_1 = M_{12}, \text{ and } M_2 = -M_{21} \tag{177}$$

Equivalently, for the relative rigid body rotation of the beam $1-2$ such that $v_2 - v_1 > 0$ (consequently, for relative end-rotations such that $\theta_i > 0$), we have for the beam $1-2$ (Fig. 11)

$$\delta W_{\text{int}}^{(1-2)} = -\sum M_i \delta\theta_i = -M_1 \cdot \underbrace{(-\delta\theta_1)}_{\text{curvature}<0} - M_2 \cdot \underbrace{\delta\theta_2}_{\text{curvature}>0} \tag{178}$$

$$= -M_{12}(-\delta\theta_1) - (-M_{21})\delta\theta_2 = M_{12} \cdot \delta\theta_1 + M_{21} \cdot \delta\theta_2 \tag{179}$$

In the above result, replace $\delta\theta$ by $\delta\psi$ to compare with previous formula (Eq. 173).

3.2 Dynamic stability

Here we start to make a mechanical equivalent discrete model then we will derive the corresponding equations of motion to be time-integrated. This is the short story. The technical details are now coming so keep your sockets.

3.2.1 Discrete mechanical model

The model is usually called a Hencky-chain model. I will come back soon ...

3.2.2 Slender portal frame

We take as an application example a slender elastic frame with a point load P on the beam mid-span. The point of application of the load is not exactly the centre but a slightly on the left in order to break the symmetry. The load is increased from 0 to a value P few times higher than the corresponding static buckling load.

The time response of the frame is shown in Figures (12 and 13).

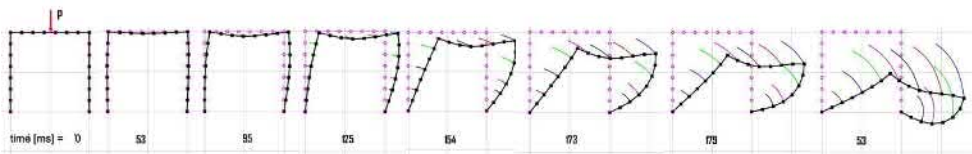


Figure 12: Hinged-hinged frame dynamic loss of stability. The load P is a bit decentred to cause breaking of symmetry. The amplitude of the load was growing from 0 to quite over the static buckling load.

3.2.3 Energy discretisation

Basic on the virtual work principle ...

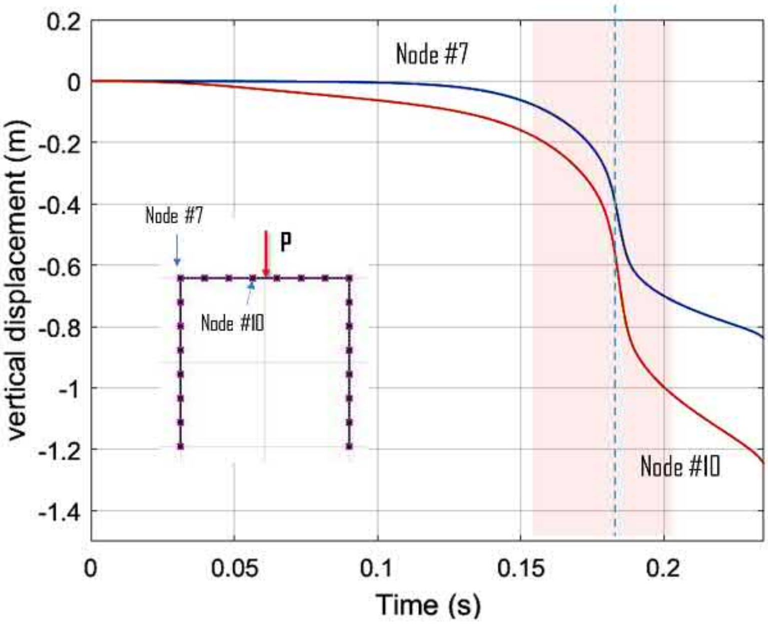


Figure 13: Hinged-hinged frame dynamic loss of stability. vertical displacement as function of time. The amplitude of the load was growing from 0 to quite over the static buckling load. IN the shaded region, the acceleration get very high brutally. This corresponds to the first instability when displacements grow suddenly too large.

4 Basic concepts and definitions

The physics: An elastic deformable body Ω is loaded by volume and surface forces, $\vec{f}$ and $\vec{t}_{\mathrm{d}}$ on the part of its surface $\Gamma_\sigma \equiv \partial\Omega_\sigma$ (Force boundary conditions). The displacements $\vec{u}$ of a part of the surface $\Gamma_{\mathrm{u}} \equiv \partial\Omega_{\mathrm{u}}$ are constrained to fixed values $\vec{u}_{\mathrm{d}}$ (Kinematic boundary conditions). The body occupies an initial configuration at time $t = 0$ and have an initial velocity. Now the basic problem is to determine the fields[38] of displacement $\vec{u}$, strains $\boldsymbol{\epsilon}$ and stresses $\boldsymbol{\sigma}$. It should be recalled that the strain and the stress fields are *tensors*. In our Cartesian space we operate mostly with the their matrix representations which are known as strain and stress matrices. *matrices* These fields are time dependent in dynamics. For static case, the time is not involved. The above setting is known as the basic problem of elasticity. For static case, just put the accelerations equal to zero.

[38]These are called *fields* (kenttiä) because they are functions defined at each point $\mathbf{x} \in \Omega \in \mathbf{R}^3$ where $\mathbf{x} = (x, y, z)$