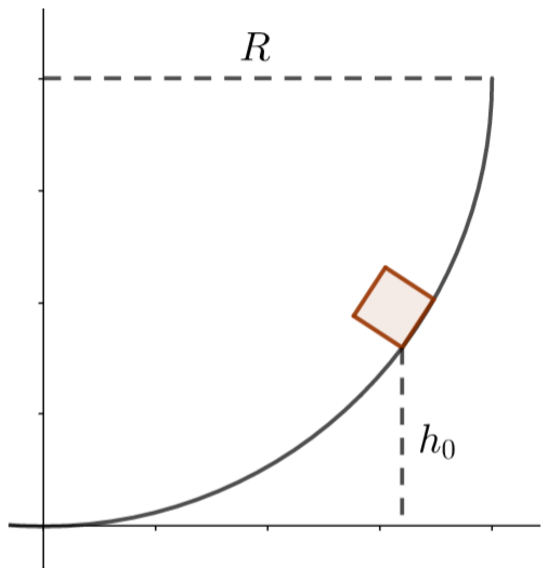
ELEC-C9420 Introduction to Quantum Technology, Fall 21
Midterm exam 1, part B, 28.10.2021
teacher: Matti Raasakka

Problem B1

An asteroid is travelling in space at speed 30 km/s. Suddenly it breaks into two parts, A and B. Part A is twice as massive as part B. The velocities of parts A and B form angles $\alpha_A = 23°$ and $\alpha_B = 15°$ with respect to the initial velocity of the original asteroid.

a) Find the speeds of the two asteroid parts after the disintegration. (6p)
b) How large was the relative change in the total mechanical energy of the system in the process? (2p)

Problem B2



A small box slides down a semi-circular ramp (radius R). The coefficient of kinetic friction between the box and the ramp is μ. At what height should the box be released from rest in order for it to stop exactly at the lowest point of the ramp? Express the initial height h_0 as a function of the ramp radius R and the coefficient of kinetic friction μ. (8p)

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Midterm exam 1, part B, model solutions

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Problem B1

An asteroid is travelling in space at speed 30 km/s. Suddenly it breaks into two parts, A and B. Part A is twice as massive as part B. The velocities of parts A and B form angles $\alpha_A = 23°$ and $\alpha_B = 15°$ with respect to the initial velocity of the original asteroid.

a) Find the speeds of the two asteroid parts after the disintegration. (6p)
b) How large was the relative change in the total mechanical energy of the system in the process? (2p)

Model solution

a) Total momentum is conserved in the process, since there are no external forces acting on the system. (1p) The total mass is also conserved, so if the mass of the original asteroid is M, then part A's mass is $\frac{2M}{3}$ and part B's mass $\frac{M}{3}$. (1p) Let's draw a diagram of the initial and final states of the system (fig. 1).

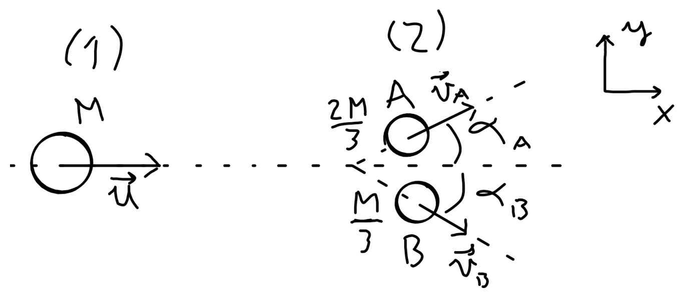


Figure 1: The system of problem B1.

From the conservation of momentum x-component we get the equation

$$Mu = \frac{2M}{3} v_A \cos\alpha_A + \frac{M}{3} v_B \cos\alpha_B \,. \quad \text{(1p)}$$

From the conservation of momentum y-component we get the equation

$$0 = \frac{2M}{3} v_A \sin\alpha_A - \frac{M}{3} v_B \sin\alpha_B \,. \quad \text{(1p)}$$

From these two equations we can solve for the final speeds v_A and v_B. From the second equation we get

$$v_B = \frac{2\sin\alpha_A}{\sin\alpha_B} v_A \,.$$

By substituting into the first equation we get

$$
\begin{aligned}
Mu &= \frac{2M}{3} v_A \cos\alpha_A + \frac{M}{3} \frac{2\sin\alpha_A \cos\alpha_B}{\sin\alpha_B} v_A \\
3u &= 2v_A \cos\alpha_A + \frac{2\sin\alpha_A}{\tan\alpha_B} v_A = 2\left(\cos\alpha_A + \frac{\sin\alpha_A}{\tan\alpha_B}\right) v_A \\
\Rightarrow \quad v_A &= \frac{u}{\frac{2}{3}\left(\cos\alpha_A + \frac{\sin\alpha_A}{\tan\alpha_B}\right)} = \frac{30\ \text{km/s}}{\frac{2}{3}\left(\cos 23^\circ + \frac{\sin 23^\circ}{\tan 15^\circ}\right)} = 18.91763\ldots\ \text{km/s} \approx 18.9\ \text{km/s}\,. \quad \text{(1p)}
\end{aligned}
$$

For the speed of part B we get

$$
\begin{aligned}
v_B &= \frac{2\sin\alpha_A}{\sin\alpha_B} \frac{u}{\frac{2}{3}\left(\cos\alpha_A + \frac{\sin\alpha_A}{\tan\alpha_B}\right)} = \frac{u}{\frac{1}{3}\left(\frac{\sin\alpha_B}{\tan\alpha_A} + \cos\alpha_B\right)} \\
&= \frac{30\ \text{km/s}}{\frac{1}{3}\left(\frac{\sin 15^\circ}{\tan 23^\circ} + \cos 15^\circ\right)} = 57.11873\ldots\ \text{km/s} \approx 57.1\ \text{km/s}\,. \quad \text{(1p)}
\end{aligned}
$$

b) The mechanical energy of the system consists only of kinetic energy. The change in the total kinetic energy during the process:

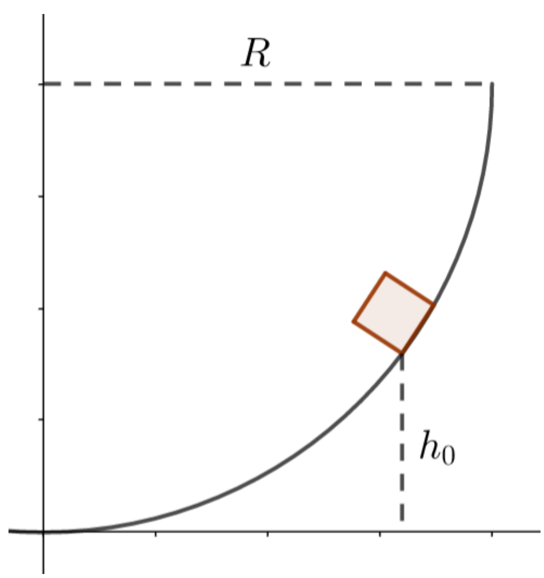
$$
\Delta K = \frac{1}{2}\frac{2M}{3} v_A^2 + \frac{1}{2}\frac{M}{3} v_B^2 - \frac{1}{2} M u^2\,. \quad \text{(1p)}
$$

The relative change is therefore given by

$$
\frac{\Delta K}{K_0} = \frac{\frac{1}{2}\frac{2M}{3} v_A^2 + \frac{1}{2}\frac{M}{3} v_B^2 - \frac{1}{2} M u^2}{\frac{1}{2} M u^2} = \frac{2}{3}\left(\frac{v_A}{u}\right)^2 + \frac{1}{3}\left(\frac{v_B}{u}\right)^2 - 1 \approx 0.473\,.
$$

Thus, the mechanical energy increased by 47.3%. (1p)

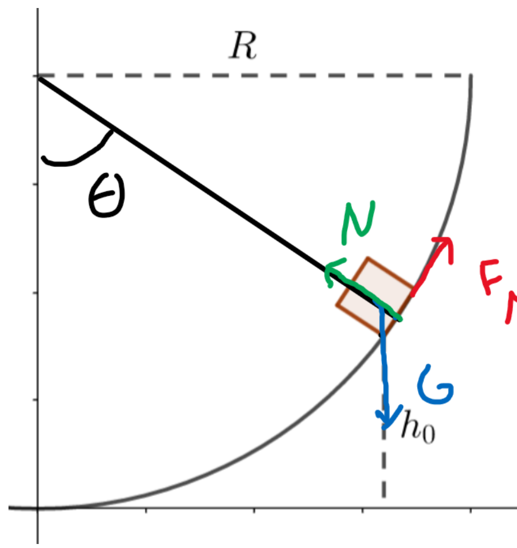
Problem B2



A small box slides down a semi-circular ramp (radius R). The coefficient of kinetic friction between the box and the ramp is μ. At what height should the box be released from rest in order for it to stop exactly at the lowest point of the ramp? Express the initial height h_0 as a function of the ramp radius R and the coefficient of kinetic friction μ. (8p)

Model solution

During the sliding the kinetic friction dissipates away all the initial gravitational potential energy of the box. (1p) The gravitational potential energy in the beginning is $U_{\text{grav}} = mgh_0$. (1p) Let's denote the angle of the line connecting the box to the center of the (semi-)circular ramp with respect to vertical by θ. Since the box only moves tangentially with respect to the ramps



surface, the normal force of the ramp must cancel out the component of the gravitational force perpendicular to the ramp, which gives $N = mg\cos\theta$. (1p) The kinetic friction is therefore given by $F_\mu(\theta) = \mu mg\cos\theta$. (1p) Let's denote the initial angle at the starting location of the box by θ_0. As the box slides down the ramp the kinetic friction does work

$$W_\mu = -\int_{\theta_0}^{0} F_\mu R \mathrm{d}\theta = \mu mgR\sin\theta_0 \,. \quad \text{(1p)}$$

By using the trigonometric identity $\sin\theta = \sqrt{1-\cos^2\theta}$ and noticing that $R\cos\theta_0 = R - h_0$ we can express the work in terms of h_0 instead of θ_0:

$$\begin{aligned} W_\mu &= \mu mgR\sqrt{1-\cos^2\theta_0} \\ &= \mu mg\sqrt{R^2 - R^2\cos^2\theta_0} \\ &= \mu mg\sqrt{R^2-(R-h_0)^2} \\ &= \mu mg\sqrt{h_0(2R-h_0)} \,. \end{aligned}$$

Therefore, we get from the conservation of energy (1p)

$$\begin{aligned} mgh_0 &= \mu mg\sqrt{h_0(2R-h_0)} \\ h_0 &= \mu\sqrt{h_0(2R-h_0)} \\ h_0^2 &= \mu^2 h_0(2R-h_0) \\ \big((1+\mu^2)h_0 - 2\mu^2 R\big)h_0 &= 0 \,. \end{aligned}$$

Accordingly, the initial height of the box must satisfy

$$h_0 = 0 \quad \text{or} \quad (1+\mu^2)h_0 - 2\mu^2 R = 0 \quad \Leftrightarrow \quad h_0 = \frac{2\mu^2}{1+\mu^2}R \,. \quad \text{(2p)}$$