CIV-E1060 Engineering Computation and Simulation Examination, December 13, 2021 / Niiranen

This examination consists of 3 problems, each rated by the standard scale 1...6.

Problem 1

(i) The definition of *derivative* for a one-variable scalar function $f = f(x)$ at point x reads as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(1) Write down the corresponding *forward finite difference* serving as an approximation for the derivative.
(2) For the polynomial function $f(x) = 2x(2-x)$, calculate first the exact value of the derivative at $x = 0$, and then, by using the same function as an example, show how decreasing the step size h increases the accuracy of the forward finite difference.
Hint: You can either use a general step size h or just two numerical example step sizes, e.g., $h = 1/2$ and $h = 1/4$.

(ii) Let us consider a *diffusion–convection–reaction* process modelled by using a one-dimensional model associated to the governing differential equation

$$-(ku')'(x) + cu'(x) + ru(x) = s(x), \quad x \in (0, L),$$

with a distributed source s and length L as well as the diffusivity (k), convection velocity (c) and reactivity (r) parameters. At one end of the problem domain (say, at $x = 0$), the primary problem variable is fixed to zero, whereas at the other end (say, at $x = L$) the flux of the primary variable is known.

In order to find an approximate solution to the boundary value problem formed by this differential equation and these boundary conditions, assume constant model parameters and utilize the *second-order finite difference*

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

together with the appropriate *first-order finite difference(s)* obtained from item (i) for writing down the matrix form system equation of the *finite difference method* corresponding to a uniform three-point grid: $x_0 = 0, x_1 = L/2, x_2 = L$.
Note: Solving the equation system is not required.

(iii) When using *quadratic* (second-order) *Lagrange* basis functions in the *finite element method* for the problems of extension-compression *bar model*, (1) which type of *numerical integration* scheme(s) should be used for evaluating the entries of the corresponding *stiffness matrix* and (2) why?

Problem 2

(i) Briefly address the key reasons (at least four) for the triumph of the *finite element method* in engineering computation and simulation.
Hint: You can consider first (1) the reasons stemming from the nature and features of the method compared to the other methods studied on this course and then (2) the reasons important for engineering modelling and computation in general (with respect to both theory and practice).

(ii) Briefly explain, preferably with the aid of some key formulae, what is meant by the so-called *reference element* technique in the context of the *finite element method*.
Hint: You can focus on either 1D or 2D formulations.

(iii) Let us consider *stationary heat conduction* in a quadrangular domain (say, simply 1 m (or a m) wide and 2 m (or b m, with $b > a$) long). The physical problem can be modeled by relying on the *first law of thermodynamics* combined with the stationary state assumption, implying the partial differential equation

$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } \Omega$$

with the *Fourier law* building a constitutive relation between heat flux $\boldsymbol{q} = (q_1(x, y), q_2(x, y))$ and temperature $T = T(x, y)$ through thermal conductivity $k = k(x, y)$ written in the form

$$\boldsymbol{q} = -k\nabla T \quad \text{in } \Omega.$$

By applying the *finite element method* of *bilinear quadrangular* elements, find a rough approximation for the peak value of the temperature field in the following simple problem set up: (a) along the boundary of the problem domain, heat is fixed to a constant value; (b) conductivity is constant; (c) heat source distribution is uniform.
Hint: You can use as many or as few elements as you consider enough, or reasonable, for the given task.

Problem 3

Let us use the *finite element method* of the *Hermite* elements for the following *Euler–Bernoulli beam* bending problem (length $2L$):

- there is no distributed loading;
- the bending rigidity is constant;
- one end of the beam (say, at $x = 0$) is clamped;
- there is a simple support in the middle of the beam (at $x = L$);
- the other end (say, at $x = 2L$) moves in the transversal direction, i.e., in the direction perpendicular to the central axis of the beam, but the rotation at this end point is prevented by a roller support rigidly joined to beam but sliding along a rail crossing the end point in the transversal direction
- there is a point load at the roller support end, acting in the transversal direction.

(1) Find a finite element deflection approximation for the problem.
(2) Briefly explain, how to obtain the corresponding moment and shear force approximations.
Hint: You can use as many or as few elements as you consider enough, or reasonable, for the given task.
Note: For constructing the finite element equation system for the problem, you can either utilize the appropriate stiffness matrix and force vector entries found in the course text book or you can construct the stiffness matrix and force vector entries by yourself by utilizing the *Hermite* basis functions given below for a generic line segment $(0, 1)$:

$$\begin{aligned}
\phi_1(x) &= 1 - 3x^2 + 2x^3, \\
\phi_2(x) &= x - 2x^2 + x^3, \\
\phi_3(x) &= 3x^2 - 2x^3, \\
\phi_4(x) &= -x^2 + x^3.
\end{aligned}$$