CIV-E1060 Engineering Computation and Simulation Examination, December 12, 2017 / Niiranen

This examination consists of 3 problems rated by the standard scale 1...6.

Problem 1

Let us consider a long and tall wall with a constant thickness L and constant temperature values on inner and outer surfaces.

If a heat source inside the wall can be modelled by a quadratic function

$$f = f(x) = f_0 x(1 - x/L),$$

where x denotes the coordinate along a line across the wall and f_0 is a constant, the temperature distribution inside the wall can be modelled by a one-dimensional stationary heat diffusion problem through the thickness of the wall: for given thermal conductivity k, heat source f and boundary temperature values T_0 and T_L, find the temperature distribution T such that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}\big(k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\big) &= f(x) \quad \forall x \in (0, L) \\ T(0) &= T_0 \\ T(L) &= T_L. \end{aligned}$$

Above, it has been assumed that the *Fourier* law builds a constitutive relation between heat flux q and temperature T through thermal conductivity k as

$$q(x) = -k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}.$$

(i) Derive the weak (variational) form of the boundary value problem.

(ii) In order to construct an approximate finite element trial function for the temperature distribution, let us devide the line interval $(0, L)$ into two elements of equal size. Write down the expressions for corresponding piecewise linear basis functions in terms of the global x-coordinate.

(iii) Calculate the analytical solution of the heat diffusion problem by assuming a constant conductivity $k = k_0$ and temperature values $T_0 = 0, T_L = 20$.

Model solutions for Problem 1 (somewhat more comprehensive and detailed than required for the maximum grade):

(i) total 2 p. First, the strong form is multiplied by a test function $v = v(x)$:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\big(k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\big)v(x) = f(x)v(x).$$

Second, the equation is integrated over the problem interval $(0, L)$:

$$-\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\big(k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\big)v(x)\,\mathrm{d}x = \int_0^L f(x)v(x)\,\mathrm{d}x$$

Third, the left hand side is integrated by parts giving

$$-\Big[k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}v(x)\Big]_0^L + \int_0^L k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L f(x)v(x)\,\mathrm{d}x.$$

1 p.

Since the essential boundary conditions $T(0) = T_0, T(L) = T_L$ are given in the strong form, the corresponding zero boundary conditions $v(0) = 0, v(L) = 0$ are set for the test function:

$$\int_0^L f(x)v(x)\,\mathrm{d}x = \int_0^L k(x)T'(x)v'(x)\,\mathrm{d}x.$$

This gives us the weak form of the problem: Find $T = T(x)$ satisfying $T(0) = T_0, T(L) = T_L$ such that

$$\int_0^L k(x)T'(x)v'(x)\,\mathrm{d}x = \int_0^L f(x)v(x)\,\mathrm{d}x$$

for all $v = v(x)$ satisfying $v(0) = 0 = v(L)$.
1 p.

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It can be noticed that the trial and test functions have to satisfy the regularity conditions

$$\int_0^L (T'(x))^2\,\mathrm{d}x < \infty, \quad \int_0^L (v'(x))^2\,\mathrm{d}x < \infty.$$

(ii) total 2 p. With two linear elements of size $h = L/2$, meaning three basis functions ϕ_i related to the nodes $x_i = 0, L/2, L$ and three degrees of freedom d_i with $i = 0, 1, 2$, the corresponding finite element approximation can be written in the form

$$T_h(x) = \sum_{i=0}^{2} \phi_i(x)d_i.$$

Lagrange basis functions satisfy the node value condition $\phi_i(x_j) = \delta_{ij}$ ($\phi_i(x_j) = 1$ for $j = i$ and $\phi_i(x_j) = 0$ whenever $j \neq i$) implying that the (unknown) degrees of freedom are actually nodal values of the temperature approximation, i.e., $d_i = T_h(x_i)$, and hence

$$T_h(x) = \sum_{i=0}^{2} \phi_i(x)T_h(x_i).$$

1 p.

The 1D tent-shaped piecewise linear basis functions are defined as follows (Draw the corresponding graphs!):

$$\phi_0(x) = \begin{cases} 1 - 2x/L, & x \in [0, L/2) \\ 0, & x \in [L/2, L] \end{cases}$$
$$\phi_1(x) = \begin{cases} 2x/L, & x \in [0, L/2) \\ 2 - 2x/L, & x \in [L/2, L] \end{cases}$$
$$\phi_2(x) = \begin{cases} 0, & x \in [0, L/2) \\ -1 + 2x/L, & x \in [L/2, L]. \end{cases}$$

1 p.

(iii) **total 2 p.** The exact solution of the problem can be solved by first inserting the constant conductivity and quadratic loading into the differential equation and then integrating the equation twice:

$$\begin{aligned} -k_0 T''(x) &= f_0 x(1 - x/L) \\ \Rightarrow T''(x) &= \frac{f_0}{k_0}(\frac{x^2}{L} - x) \\ \Rightarrow T'(x) &= \frac{f_0}{k_0}(\frac{x^3}{3L} - \frac{x^2}{2}) + c_1 \\ \Rightarrow T(x) &= \frac{f_0}{k_0}(\frac{x^4}{12L} - \frac{x^3}{6}) + c_1 x + c_2 \end{aligned}$$

1 p.

The integration constants c_1 and c_2 can be solved from the essential boundary conditions:

$$\begin{aligned} 0 &= T_0 = T(0) = c_2 \\ 20 &= T_L = T(L) = -\frac{f_0}{k_0}\frac{L^3}{12} + c_1 L \Rightarrow c_1 = 20/L + \frac{f_0 L^2}{12 k_0}. \end{aligned}$$

The exact solution is finally written as

$$\begin{aligned} T(x) &= \frac{f_0}{k_0}(\frac{x^4}{12L} - \frac{x^3}{6}) + (20/L + \frac{f_0 L^2}{12k_0})x \\ &= \frac{f_0 L^3}{12k_0}(x/L)^4 - \frac{f_0 L^3}{6k_0}(x/L)^3 + (20 + \frac{f_0 L^3}{12k_0})(x/L) \end{aligned}$$

showing, in particular, that conditions $T(0) = 0, T(L) = 20$ are satisfied.
1 p.

Problem 2

Let us consider stationary heat conduction in the wall of Problem 1 by using a two-dimensional model over the quadrangular cross section of the wall with H denoting the height of the wall. The bottom and top surfaces of the wall are assumed to be insulated, i.e., the heat flux vanishes along the boundary lines $y = 0$ and $y = H$ with y denoting the vertical coordinate of the cross section.

(i) For a given isotropic thermal conductivity k, heat source distribution f and boundary temperature values T_0 and T_L, formulate the strong form of the two-dimensional boundary value problem (draw a picture as well).

(ii) Describe how to derive a finite element approximation for the solution of the problem, including information about a triangular discretization with linear Lagrange basis functions, as well as numerical integration and assembly of the corresponding finite element matrices and vectors.

Model solutions for Problem 3 (somewhat more comprehensive and detailed than required for the maximum grade):

(i) **total 2 p.** The strong form of the problem is derived by simply imitating the 1D formulation of Problem 1 (by simply replacing x with (x, y), and accordingly $T(x)$ with $T(x, y)$, $k(x)$ with $k(x, y)$, $f(x)$ with $f(x, y)$ and d/dx with ∇): for given $k = k(x, y)$, $f = f(x, y)$ and T_0, T_L, find $T = T(x, y)$ such that

$$\begin{aligned}
-\nabla \cdot (k\nabla T)(x, y) &= f(x, y), \quad (x, y) \in \Omega, \\
T(x, y) &= T_0, \quad (x, y) \in \Gamma_{T_0} \subset \partial\Omega, \\
T(x, y) &= T_L, \quad (x, y) \in \Gamma_{T_L} \subset \partial\Omega, \\
-(k\nabla T \cdot \boldsymbol{n})(x, y) &= q_0 = 0, \quad (x, y) \in \Gamma_q \subset \partial\Omega,
\end{aligned}$$

where $\Omega = (0, L) \times (0, H) \subset \mathbb{R}^2$ denotes the problem domain (the rectangular cross section of the wall), $\boldsymbol{n}$ denotes the outward normal of the boundary curve $\partial\Omega$ (the edge lines of the rectangle), $\Gamma_{T_0} = \{\{x = 0\} \times (0, H)\}$ and $\Gamma_{T_L} = \{\{x = L\} \times (0, H)\}$ stand for the boundary parts of the essential boundary conditions (the vertical left and right edges) and $\Gamma_q = \{(0, L) \times \{y = 0\}\} \cup \{(0, L) \times \{y = H\}\}$ defines the boundary parts of the natural boundary condition (the horizontal bottom and top edges) and q_0 denotes the given heat flux across the boundary part Γ_q.

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I can be noticed that the divergence operation can be written as

$$\nabla \cdot (k\nabla T)(x, y) = \frac{\mathrm{d}}{\mathrm{d}x}\big(k(x, y)\frac{\mathrm{d}T(x, y)}{\mathrm{d}x}\big) + \frac{\mathrm{d}}{\mathrm{d}y}\big(k(x, y)\frac{\mathrm{d}T(x, y)}{\mathrm{d}y}\big).$$

(ii) **total 4 p.** The integral equation of the weak form of the problem is achieved as usual: multiplying with a test function $v = v(x, y)$, integrating over the domain and integrating by parts (once) giving

$$\int_\Omega (k\nabla T)(x,y)\cdot\nabla v(x,y)\,\mathrm{d}\Omega - \Big(\int_{\Gamma_{T_0}} + \int_{\Gamma_{T_L}} + \int_{\Gamma_{T_q}}\Big)(k\nabla T)(x,y)\cdot\boldsymbol{n}\,v(x,y)\,\mathrm{d}s$$
$$= \int_\Omega f(x,y)\,v(x,y)\,\mathrm{d}\Omega$$
$$\Rightarrow \quad \int_\Omega (k\nabla T)(x,y)\cdot\nabla v(x,y)\,\mathrm{d}\Omega = \int_\Omega f(x,y)\,v(x,y)\,\mathrm{d}\Omega.$$

1 p.

The element stiffness matrix entries k^e_{ij}, $i,j = 1,2,3$, and the force vector entries f_i are calculated according to the weak form:

$$k^e_{ij} = \int_e (k\nabla\phi^e_i)(x,y)\cdot\nabla\phi^e_j(x,y)\,\mathrm{d}x\mathrm{d}y, \quad f_i = \int_e f(x,y)\,\phi^e_i(x,y)\,\mathrm{d}x\mathrm{d}y,$$

where $\phi^e_i = \phi^e_i(x,y)$ are the shape functions of the element. Typically, these xy-integrals over elements are transformed (with affine mappings) onto the reference element (and integrands as well with appropriate transformations with Jacobian matrices) and then numerically integrated over the reference element.

1 p.

Let us focus on one element of the mesh and, in particular, on its contribution to the global stiffness matrix of the final finite element system equation. The chosen three-node element situates "in the origin"of the xy-coordinate system with corner points $(0,0),(h,0),(0,h)$.

The local stiffness matrix of the chosen element (as any other element) is typically calculated by using a reference element of the $\xi\eta$-coordinate system with corner points $(0,0),(1,0),(0,1)$. For the first order (linear) triangular reference element, the 2D Lagrange shape functions $N_i(\xi,\eta) = A_i\xi + B_i\eta + C_i$ – associated to corner points $c_1 = (0,0), c_2 = (1,0), c_3 = (0,1)$ of the $\xi\eta$-coordinate system – are

$$N_1(\xi,\eta) = 1-\xi-\eta, \quad N_2(\xi,\eta) = \xi, \quad N_3(\xi,\eta) = \eta$$

since they satisfy conditions $N_i(c_j) = \delta_{ij}$, $i,j = 1,2,3$ (meaning that $N_1(c_1) = 1$ but $N_1(c_2) = 0 = N_1(c_3)$ and so on).

1 p.

The global stiffness matrix $\boldsymbol{K}$ of the system $\boldsymbol{Kd} = \boldsymbol{f}$ (and the force vector $\boldsymbol{f}$ correspondingly) is assembled such that each entry in the global stiffness matrix (corresponding to one node) gets contributions from the corresponding entries of the local 3×3 stiffness matrices $\boldsymbol{K}^e$ of elements sharing the node of the global entry.

If the (x,y)-origin lies in one corner of the domain occupied by the chosen element alone (as in this case), it holds that $k_{(0,0)} = k^e_{11}$. (Draw a picture!)

If the origin situates inside the domain and four elements surround it, for instance – say, e with corner points $(0,0),(h,0)$ and $(0,h)$, e' with corner points $(0,0),(0,-h)$ and $(h,0)$, e'' with corner points $(0,0),(-h,0)$ and $(0,-h)$ and e''' with corner points $(0,0),(0,h)$ and $(-h,0)$ – the corresponding entry in the global stiffness matrix is $k_{(0,0)} = k_{11}^{e} + k_{11}^{e'} + k_{11}^{e''} + k_{11}^{e'''}$. (Draw a picture!)
1 p.

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I can be noticed that since the chosen example element e can be obtained by simply scaling the reference element by h in both coordinate directions (Draw a picture!), the shape functions in global coordinates can be obtained by the simple change of variables, $\xi = x/h, \eta = y/h$, giving

$$\phi_1(x,y) = 1 - x/h - y/h, \quad \phi_2(x,y) = x/h, \quad \phi_3(x,y) = y/h$$

and the corresponding gradients

$$\nabla\phi_1(x,y) = \begin{pmatrix} -1/h \\ -1/h \end{pmatrix}, \quad \nabla\phi_2(x,y) = \begin{pmatrix} 1/h \\ 0 \end{pmatrix}, \quad \nabla\phi_3(x,y) = \begin{pmatrix} 0 \\ 1/h \end{pmatrix}.$$

For these constant values (together with assuming constant conductivity k_0), the stiffness coefficients reduce – without numerical integration – to

$$k_{ij}^{e} = \int_{e} (k\nabla\phi_i^e)(x,y)\cdot\nabla\phi_j^e(x,y)\,\mathrm{d}x\mathrm{d}y = k_0\nabla\phi_i^e\cdot\nabla\phi_j^e\,|e| = k_0\nabla\phi_i^e\cdot\nabla\phi_j^e\,h^2/2$$

giving element stiffness matrix finally in the form

$$\boldsymbol{K}^e = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} = \frac{k_0}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Problem 3

The governing equation of the Euler–Bernoulli beam problem can be written in the classical form

$$EIw'''' = f.$$

(i) Define and name the quantities, variables and other notation appearing in the given formulation.

(ii) Derive the weak form of the problem corresponding to a cantilever beam and write down the corresponding conforming finite element formulation.

(iii) Form the finite element equation system for a uniformly loaded cantilever beam by adopting one single element relying on Hermite basis functions

$$\begin{aligned}
\phi_1(x) &= 1 - 3x^2 + 2x^3, \\
\phi_2(x) &= x - 2x^2 + x^3, \\
\phi_3(x) &= 3x^2 - 2x^3, \\
\phi_4(x) &= -x^2 + x^3.
\end{aligned}$$

Model solutions for Problem 3 (somewhat more comprehensive and detailed than required for the maximum grade):

(i) **total 1 p.** EI denotes the bending rigidity of the beam (assumed to be constant in the given form of the differential equation but in general $EI = E(x)I(x)$), where Young's modulus (or elastic modulus) E expresses the stiffness of the material and I denotes the area moment of inertia (or second moment of area) of the cross section of the beam; $w = w(x)$ (a C^4-continuous function in principle) denotes the deflection of the neutral axis of the beam with x denoting the coordinate along the neutral axis; $w' = w'(x) = dw(x)/dx$ denotes the x-derivative of the deflection; $f = f(x)$ denotes the distributed external loading acting transversally on the beam (and consisting of both body loads as dead weight and surface tractions).

(ii) **total 3 p.** As always, for deriving the weak form of the problem (and finally the full strong form as well), first, the differential equation is multiplied by a test function $\tilde{w} = \tilde{w}(x)$ and then integrated over the interval:

$$EIw''''(x)\tilde{w}(x) = f(x)\tilde{w}(x).$$

Second, both sides of the equation are integrated over the problem domain:

$$\int_0^L EIw''''(x)\tilde{w}(x)dx = \int_0^L f(x)\tilde{w}(x)dx.$$

Third, applying integration by parts in the left hand side yields

$$EIw'''(x)\tilde{w}(x)|_0^L - \int_0^L EIw'''(x)\tilde{w}'(x)dx = \int_0^L f(x)\tilde{w}(x)dx.$$

Fourth, applying integration by parts once again yields

$$EIw'''(x)\tilde{w}(x)|_0^L - EIw''(x)\tilde{w}'(x)|_0^L + \int_0^L EIw''(x)\tilde{w}''(x)dx = \int_0^L f(x)\tilde{w}(x)dx.$$

1 p.

This form, actually the substitution terms

$$\begin{aligned} &EIw'''(x)\tilde{w}(x)|_0^L - EIw''(x)\tilde{w}'(x)|_0^L \\ &= EIw'''(L)\tilde{w}(L) - EIw'''(0)\tilde{w}(0) - (EIw''(L)\tilde{w}'(L) - EIw''(0)\tilde{w}'(0)), \end{aligned}$$

already reveals the essential and natural boundary conditions of the problem: either (shear force) EIw''' is given (natural bc) or (deflection) w is given (essential bc); either (bending moment) EIw'' is given (natural bc) or (rotation) w' is given (essential bc).

In a cantilever beam, one end is clamped, while the other end is free (or possibly a given point load or point moment is given). Taking into account the natural (force) boundary conditions $Q(L) = -EIw'''(L) = Q_L = 0, M(L) = -EIw''(L) = M_L = 0$ (both assumed to be zero for simplicity) of the free (right) end and setting boundary conditions $\tilde{w}(0) = 0, \tilde{w}'(0) = 0$ corresponding to the essential (displacement) boundary conditions $w(0) = 0, w'(0) = 0$ of the clamped (left) end "kills"the substitution terms (or makes them known if $Q_L \neq 0$ or $M_L \neq 0$) and finally gives the weak form of the problem: find $w \in H^2(0, L)$ such that $w(0) = 0 = w'(0)$ and

$$\int_0^L EIw''(x)\tilde{w}''(x)dx = \int_0^L f(x)\tilde{w}(x)dx$$

for all $\tilde{w} \in H^2(0, L)$ satisfying conditions $\tilde{w}(0) = 0 = \tilde{w}'(0)$.
1 p.

The corresponding conforming finite element formulation reads as follows: find $w_h \in H^2(0, L)$ such that $w_h(0) = 0 = w_h'(0)$, $w_{h|K} \in P_k(K)$ in each element K (w_h is a piece-wise polynomial function of order k) and

$$\int_0^L EIw_h''(x)\tilde{w}''(x)dx = \int_0^L f(x)\tilde{w}(x)dx$$

for all $\tilde{w} \in H^2(0, L)$, $\tilde{w}_{|K} \in P_k(K)$, satisfying conditions $\tilde{w}(0) = 0 = \tilde{w}'(0)$. In practice, $w_h \in C^1(0, L)$ should hold and for constructing a C^1-continuous approximation polynomial order $k = 3$ is enough if Hermite basis functions ϕ_i are adopted: $w_h(x) = \sum_i \phi_i(x)d_i$.
1 p.

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It can be noticed that the strong form of the Euler–Bernoulli beam bending problem for a cantilever beam can be written in the form

$$
\begin{aligned}
EIw''''(x) &= f(x), \quad x \in (0, L), \\
w(0) &= w_0 = 0, \\
w'(0) &= \beta_L = 0, \\
Q(L) = -EIw'''(L) &= Q_L = 0, \\
M(L) = -EIw''(L) &= M_L = 0.
\end{aligned}
$$

(iii) **total 2 p.** The given Hermite basis functions, correspond to one single element with end points $x = 0$ and $x = L = 1$ since $\phi_1(0) = 0$, $\phi_2'(0) = 0$, $\phi_3(0) = 0$, $\phi_4'(0) = 0$. Accordingly, basis functions ϕ_1 and ϕ_2 correspond to the degrees of freedom $d_1 = w_h(0)$ and $d_2 = w_h'(0)$, respectively, whereas ϕ_3 and ϕ_4 correspond to the degrees of freedom $d_3 = w_h(L)$ and $d_4 = w_h'(L)$, respectively. The finite element approximation is then of the form

$$
\begin{aligned}
w_h(x) &= \sum_{i=1}^{4} \phi_i(x) d_i \\
&= \phi_1(x) w_h(0) + \phi_2(x) w_h'(0) + \phi_3(x) w_h(L) + \phi_4(x) w_h'(L) \\
&= \phi_3(x) w_h(L) + \phi_4(x) w_h'(L),
\end{aligned}
$$

where the last equality follows from the fact that the finite element approximation needs to satisfy the essential boundary conditions $d_1 = w_h(0) = 0, d_2 = w_h'(0) = 0$ (and the test function as well) expressed in the finite element formulation above in item (ii).
1 p.
The original 4×4 equation system now reduces to a 2×2 the system of two unknowns (d_3 and d_4):

$$
\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f} \quad \Leftrightarrow \quad \begin{pmatrix} k_{33} & k_{34} \\ k_{43} & k_{44} \end{pmatrix} \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}.
$$

The stiffness matrix and force vector entries take their forms from the finite element formulation as follows:

$$
k_{ij} = \int_0^1 EI\phi_i''(x)\phi_j''(x)dx = k_{ji}, \quad f_i = \int_0^1 f(x)\phi_i(x)dx,
$$

where the shape function derivatives are given as

$$
\phi_1''(x) = -6 + 12x, \ \phi_2''(x) = -4 + 6x, \ \phi_3''(x) = 6 - 12x, \ \phi_4''(x) = -2 + 6x.
$$

After calculating the stiffness matrix entries $k_{33}, k_{34} = k_{43}, k_{44}$ (for constant bending rigidity EI, it holds that $k_{33} = 12EI, k_{34} = -6EI = k_{43}, k_{44} = 4EI$) and force vector components f_3, f_4 for bending stiffness EI and loading f (not prescribed in the problem setting), the equation system, and finally the finite element approximation for the deflection, can be easily solved: $\boldsymbol{d} = \boldsymbol{K}^{-1}\boldsymbol{f}$.

CIV-E1060 Engineering Computation and Simulation Examination, December 11, 2018 / Niiranen

This examination consists of 3 problems, each rated by the standard scale 1...6.

Problem 1

The derivative of function $f = f(x)$ at point x is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

(i) Write down the following three approximations for this derivative: forward finite difference, backward finite difference and central finite difference.

(ii) What are the main differences between the three approximations?

(iii) Let us consider the extension–compression state of a longish structure modelled by using the engineering rod model implying the governing differential equation

$$-(EAu')'(x) = b(x), \quad x \in (0, L),$$

with axial distributed loading b, structural length L as well as constant cross-sectional area A and stiffness modulus E. One end of the bar structure is fixed, whereas the other end is loaded by an axial endpoint force N_L, implying boundary conditions

$$u(0) = 0$$
$$EAu'(L) = N_L.$$

In order to find an approximate solution to the boundary value problem formed by the differential equation and boundary conditions, utilize the second-order finite difference approximation

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

together with the finite difference approximations of item (i) for writing down the system equations of the finite difference method corresponding to a uniform three-point grid: $x_0 = 0, x_1 = L/2, x_2 = L$.

Model solutions for Problem 1

(i) The forward finite difference follows the definition of the derivative:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The backward finite difference can be derived from the forward one by setting $x + h = z$ which gives $x = z - h$ and finally replacing z by x):

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}.$$

The central finite difference is the average of these two:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Drawing a picture where all these differences approximate the true derivative tells a lot. **3 p.**

(ii) Each approximation requires two function evaluations (in the first two, at the point itself and at a point either a bit ahead or a bit behind; in the last one, at points a bit behind and a bit ahead) subtraction and division but the last one is more accurate (of order h^2) than the other two (of order h). **1 p.**

(iii) The essential and natural boundary conditions give two equations, respectively:

$$\begin{aligned} 0 =& u(0) = u(x_0) \\ N_L =& EAu'(L) = EAu'(x_2) \approx EA\frac{u(x_2) - u(x_1)}{L/2}. \end{aligned}$$

The differential equation gives equation

$$b(x_1) = -EAu''(x) \approx -EA\frac{u(x_0) - 2u(x_1) + u(x_2)}{(L/2)^2}$$

Together these equations give $u(x_0) = 0$ and the system

$$\begin{aligned} -EA\frac{-2u(x_1) + u(x_2)}{L^2/4} &= b(x_1) \\ EA\frac{u(x_2) - u(x_1)}{L/2} &= N_L \end{aligned}$$

or in the matrix form

$$EA\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u(x_1) \\ u(x_2) \end{bmatrix} = \begin{bmatrix} b(x_1)L^2/4 \\ N_L L/2 \end{bmatrix}.$$

2 p.

Problem 2

Let us consider a longish structure supported as follows: one end is clamped and the other end is simply supported. The beam-like structure is subject to distributed transversal and axial loadings acting along the central axis and to an axial endpoint force and a bending moment acting at the simply supported end.

Let us assume that the bending state of the structure is modelled by using the Euler–Bernoulli beam theory with linear strains and linearly elastic material properties implying the governing equation

$$(EIw'')''(x) = f(x), \quad x \in (0, L),$$

whereas the extension–compression state of the structure is modelled by using the engineering rod model with linear strains and linearly elastic material properties implying the governing equation

$$-(EAu')'(x) = b(x), \quad x \in (0, L).$$

(i) Write down the complete strong forms of both boundary value problems corresponding to the given differential equations and to the given description of the physical problem setting.

(ii) The integral equation in the weak form of the engineering rod extension–compression problem is written as

$$\int_0^L EAu'(x)v'(x)\,\mathrm{d}x = \int_0^L b(x)v(x)\,\mathrm{d}x + N_L\,v(L).$$

Derive the integral equation of the weak form of the Euler–Bernoulli beam bending problem.

(iii) Shortly describe the main differences between the weak forms of the Euler–Bernoulli beam bending problem and engineering rod problem and clarify the consequences of these differences to the corresponding finite element methods.

Model solutions for Problem 2

(i) The engineering rod problem: Find $u = u(x)$ such that

$$\begin{aligned} -(EAu')'(x) &= b(x), \quad x \in (0, L), \\ u(0) &= 0 \\ EAu'(L) &= N(L) = N_L. \end{aligned}$$

The Euler–Bernoulli beam problem: Find $w = w(x)$ such that

$$\begin{aligned} (EIw'')''(x) &= f(x), \quad x \in (0, L), \\ w(0) &= 0, w'(0) = 0 \\ w(L) &= 0, -EIw''(L) = M(L) = M_L \end{aligned}$$

2 p.

(ii) The integral equation is derived by multiplying by a test function $v = v(x)$, integrating over the interval and by integrating by parts twice:

$$(EIw'')''(x) = f(x), \quad x \in (0, L)$$
$$(EIw'')''(x)v(x) = f(x)v(x), \quad x \in (0, L)$$
$$\int_0^L (EIw'')''(x)v(x)dx = \int_0^L f(x)v(x)dx$$
$$[(EIw'')'(x)v(x)]_0^L - \int_0^L (EIw'')'(x)v'(x)dx = \int_0^L f(x)v(x)dx$$
$$[(EIw'')'(x)v(x)]_0^L - [EIw''(x)v'(x)]_0^L + \int_0^L (EIw'')(x)v''(x)dx = \int_0^L f(x)v(x)dx$$

Inserting the natural boundary condition $-EIw''(L) = M_L$ into the upper limit of the second substitution term, and setting $v(0) = 0, v'(0) = 0, v(L) = 0$ into the second substitution term give

$$\int_0^L (EIw'')(x)v''(x)dx = \int_0^L f(x)v(x)dx - M_L v'(L)$$

3 p.

(iii) The integral form of the bar problem has no more than one derivative for both functions w and v, which means that the finite element approximation must be continuous – a piecewise polynomial continuous approximation is fine (even piecewise linear, first order, basis functions are enough, for instance). The integral form of the beam problem has two derivatives for both functions w and v, which means that the finite element approximation must be C^1-continuous: even its derivative must be continuous – a piecewise polynomial C^1-continuous approximation is required (piecewise cubic, third order, Hermite-basis functions are needed, for instance).

1 p.

Problem 3

Let us consider stationary heat conduction in a plane domain. The physical problem can be modeled by relying on the first law of thermodynamics combined with the stationary state assumption implying the partial differential equation

$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } \Omega$$

with the *Fourier* law building a constitutive relation between heat flux $\boldsymbol{q} = (q_1(x, y), q_2(x, y))$ and temperature $T = T(x, y)$ through thermal conductivity $k = k(x, y)$ in the form

$$\boldsymbol{q} = -k\nabla T \quad \text{in } \Omega.$$

The integral equation in the weak form of the corresponding boundary value problem – with appropriate boundary conditions as well as trial and test functions – reads as follows:

$$\int_\Omega (k\nabla T)(x, y) \cdot \nabla v(x, y)\, \mathrm{d}\Omega = \int_\Omega f(x, y)\, v(x, y)\, \mathrm{d}\Omega \quad \forall v \in V.$$

Let us solve the corresponding problem by a finite element method with bilinear (first order) quadrangular elements. Let us focus on one element of the mesh and, in particular, on its contribution to the global stiffness matrix of the final finite element system equation.

(i) The chosen element is placed "in the origin" of the xy-coordinate system with corner points $(0, 0), (h, 0), (h, h), (0, h)$. Write down the standard first order Lagrange basis functions of the element in terms of coordinates x and y.

(ii) Typically, the local stiffness matrix of the chosen element (as any other element) is calculated by using a reference element of the $\xi\eta$-coordinate system with corner points $(-1, -1), (1, -1), (1, 1), (-1, 1)$. Write down the standard first order Lagrange basis functions of the reference element in terms of coordinates ξ and η.

(iii) Calculate the local stiffness matrix of the chosen element by using either the shape functions written in the global coordinate system or the ones written in the reference element coordinate system with the corresponding coordinate transformations. Thermal conductivity is assumed to be constant: $k = k_0$.

(iv) Describe briefly, possibly with some formulae or results from (i)–(iii), how the local element stiffness matrix contributes to the global stiffness matrix of the problem in the assembly process of a quadrangular mesh.

Model solutions for Problem 3

(i) Each basis function should have value 1 at its node and decrease bilinearly to value 0 reached at the other nodes. The shape function corresponding to node (h, h) is the easiest one:

$$\phi_3(x, y) = xy/h^2.$$

The shape function corresponding to node $(0, 0)$ is the next one:

$$\phi_1(x, y) = (x - h)(y - h)/h^2.$$

The shape function corresponding to nodes $(h, 0)$ and $(0, h)$ are mixtures of the first two, respectively:

$$\phi_2(x, y) = x(y - h)/h^2, \phi_4(x, y) = (x - h)y/h^2.$$

Each function is of the form $a+bx+cy+dxy$, with some constants a, b, c, d. **1 p**.

(ii) Each basis function should have value 1 at its node and decrease bilinearly to value 0 reached at the other nodes. The shape function corresponding to node $(1, 1)$ is the easiest one:

$$N_3(\xi, \eta) = (1 + \xi)(1 + \eta)/4.$$

The shape function corresponding to node $(-1, -1)$ is the next one:

$$N_1(\xi, \eta) = (1 - \xi)(1 - \eta)/4.$$

The shape function corresponding to nodes $(1, -1)$ and $(-1, 1)$ are mixtures of the first two, respectively:

$$N_2(\xi, \eta) = (1 + \xi)(1 - \eta)/4, N_4(\xi, \eta) = (1 - \xi)(1 + \eta)/4.$$

Each function is of the form $a+b\xi+c\eta+d\xi\eta$, with some constants a, b, c, d. **1 p**.

(iii) The stiffness matrices of every element e follow the left hand side of the the integral equation:

$$\begin{aligned}
k_{ij} &= \int_e k_0 \nabla\phi_i(x, y) \cdot \nabla\phi_j(x, y)\,\mathrm{d}\Omega \\
k_{11}^e &= ... = 2k_0/3 = k_{22}^e = k_{33}^e = k_{44}^e \\
k_{12}^e &= ... = -k_0/6 = k_{14}^e = k_{23}^e = k_{34}^e \\
k_{13}^e &= ... = -k_0/3 = k_{24}^e
\end{aligned}$$

The rest entries of the 4×4 element stiffness matrix follow from the symmetry of the matrix. **3 p**.

(iv) In the global stiffness matrix, there is one entry for each node. In a mesh of quadrangular elements, every node has four element around it. This means that every entry is a sum of four element contributions. For instance, if node number i (say, $i = 9$ in the global numbering) is surrounded by four elements such that the upper right corner of element 1, upper left corner of element 2, lower left corner of element 3 and lower right corner of element 4 meet at node i, the global diagonal stiffness entry of node i is $k_{ii} = k^1_{33} + k^2_{44} + k^3_{11} + k^4_{22}$. (The other entries corresponding to node $i = 9$, i.e., $k_{9j} = k_{j9}$ must be considered in a bit more complex manner but most of them are zeros as typical in finite element methods.)

1 p.

CIV-E1060 Engineering Computation and Simulation Examination, December 10, 2019 / Niiranen

This examination consists of 3 problems, each rated by the standard scale 1...6.

Problem 1

The derivative of function $f = f(x)$ at point x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(i) Write down the following three approximations for this derivative: forward finite difference, backward finite difference and central finite difference. Draw a picture illustrating the derivative and its three approximations.

(ii) Let us consider heat conduction modelled by using a one-dimensional model associated to the governing differential equation

$$-(kT')'(x) = s(x), \quad x \in (0, L),$$

with a distributed heat source s, structural length L and thermal conductivity k. At one end of the structure, temperature is fixed to zero, whereas at the other end heat flux $q(x) = -k(x)T'(x)$ is known, implying boundary conditions

$$\begin{aligned} T(0) &= 0 \\ -k(L)T'(L) &= q_L. \end{aligned}$$

In order to find an approximate solution to the boundary value problem formed by the differential equation and boundary conditions, utilize the second-order finite difference approximation

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

together with the finite difference approximation(s) of item (i) for writing down the system equations of the finite difference method corresponding to a uniform four-point grid: $x_0 = 0, x_1 = L/3, x_2 = 2L/3, x_3 = L$.

(iii) Briefly explain the most fundamental differences in solving the same model problem (1) by the finite difference method (as described above) and (2) by the finite element method with linear elements (say, with x_0, x_1, x_2 and x_3 as nodal points).

Model solutions for Problem 1

(i) **2 p**. The forward finite difference follows the definition of derivative by simply dropping the limit away:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The backward finite difference can be derived from the forward one by setting $x + h = z$ which gives $x = z - h$, and finally replacing z by x for convenience):

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}.$$

The central finite difference is the average of these two:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Drawing a picture (with curve $y = f(x)$) where all these differences approximate the true derivative (the tangent of the curve at x) tells a lot.

(ii) **3 p**. First, the essential and natural boundary conditions – valid at points $x_0 = 0$ and $x_3 = L$, respectively – give the first two equations:

$$0 = T(0) = T(x_0) \tag{1}$$

$$q_L = -k(L)T'(L) = -k(x_3)T'(x_3) \approx -k(x_3)\frac{T(x_3) - T(x_2)}{L/3}, \tag{2}$$

where the backward finite difference from above has been used with $x = x_3$, $x - h = x_2$ and $h = L/3$.

Second, for the differential equation involving the second order derivatives, let us first assume that k is constant (giving $(kT')' = k'T' + kT'' = kT''$). The differential equation – valid at points $x_1 = L/3$ and $x_2 = 2L/3$ – gives then two equations:

$$s(x_1) = -kT''(x_1) \approx -k\frac{T(x_0) - 2T(x_1) + T(x_2)}{(L/3)^2} \tag{3}$$

$$s(x_2) = -kT''(x_2) \approx -k\frac{T(x_1) - 2T(x_2) + T(x_3)}{(L/3)^2} \tag{4}$$

For non-constant k, the right hand sides should be augmented with the appropriate finite differences for $k'T'$ at x_1 and x_2, respectively (now omitted).

Together with the boundary conditions, these equations give the following system of equations (for constant k):

$$\begin{aligned} T(x_0) &= 0 \\ -k\frac{-2T(x_1) + T(x_2)}{L^2/9} &= s(x_1) \\ -k\frac{T(x_1) - 2T(x_2) + T(x_3)}{L^2/9} &= s(x_2) \\ -k\frac{-T(x_2) + T(x_3)}{L/3} &= q_L \end{aligned}$$

or in a matrix form

$$k \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T(x_0) \\ T(x_1) \\ T(x_2) \\ T(x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ s(x_1)L^2/9 \\ s(x_2)L^2/9 \\ -q_L L/3 \end{bmatrix}.$$

In this equation system, the order of the rows corresponds to the equations above in the order (1), (3), (4), (2).

(iii) **1 p**. In the finite difference method (as described above), the differential equation and the boundary conditions, i.e., the strong form of the problem, is forced to be true at a fixed number of grid points (x_i above) (some of them on the boundary of the solution domain, most of them inside the domain). The unknowns of the problem are the values of the primary problem variable at the grid points (as $T(x_i)$ above). There is no trial solution in this approximation method, neither a test function.

In the finite element method, instead, the strong form is transformed into the corresponding weak form which is an integral equation. The solution domain is divided into smaller elements (as line segments in 1D having nodal points as end points, cf. the grid points above). A continuous trial solution, typically a piecewise polynomial function, is formed as a sum of, typically polynomial, basis functions and unkown function values of the primary problem variable at nodal points.

Problem 2

Let us consider stationary heat conduction in a plane domain. The physical problem can be modeled by relying on the first law of thermodynamics combined with the stationary state assumption implying the partial differential equation

$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } \Omega$$

with the *Fourier* law building a constitutive relation between heat flux $\boldsymbol{q} = (q_1(x,y), q_2(x,y))$ and temperature $T = T(x,y)$ through thermal conductivity $k = k(x,y)$ in the form

$$\boldsymbol{q} = -k\nabla T \quad \text{in } \Omega.$$

The integral equation in the weak form of the corresponding boundary value problem – with appropriate boundary conditions as well as trial and test functions – reads as follows:

$$\int_\Omega (k\nabla T)(x,y) \cdot \nabla v(x,y)\,\mathrm{d}\Omega = \int_\Omega f(x,y)\,v(x,y)\,\mathrm{d}\Omega \quad \forall v \in V.$$

Let us solve the corresponding problem by a finite element method with linear (first order) triangular elements. Let us focus on one element of the mesh and, in particular, on its contribution to the global stiffness matrix of the final finite element system equation.

(i) The chosen element is placed "in the origin" of the xy-coordinate system with corner points $(0,0), (h,0), (0,h)$. Write down the standard first order Lagrange basis functions of the element in terms of coordinates x and y.

(ii) Calculate the local stiffness matrix of the chosen element by using the shape functions written in the global coordinate system. For simplicity, thermal conductivity is assumed to be constant: $k = k_0$.

(iii) Describe briefly, possibly with some formulae or results from (i)–(ii), how the local element stiffness matrix contributes to the global stiffness matrix of the problem in the assembly process of a triangular mesh.

Model solutions for Problem 2

(i) **2 p.** The Lagrange shape functions of the chosen element placed "in the origin" of the xy-coordinate system with corner points $c_1 = (0,0), c_2 = (h,0), c_3 = (0,h)$ can be obtained either (1) by first forming the corresponding functions for a reference element with corner points $(0,0), (1,0), (0,1)$ $\xi\eta$-coordinate system and then using a simple change of variables $\xi = x/h, \eta = y/h$ or by (2) working only with the global coordinates as follows:

Since the basis function are linear, they are of the form $\phi_i(x,y) = A_i + B_i x + C_i y, i = 1,2,3$, and they satisfy conditions $\phi_i(c_j) = \delta_{ij}$, $i,j = 1,2,3$, at corner points c_j. As an example, for the first basis function it holds that

$$\begin{aligned}
1 &= \phi_1(c_1) = \phi_1(0,0) = A_1,\\
0 &= \phi_1(c_2) = \phi_1(0,h) = A_1 + C_1 h\\
0 &= \phi_1(c_3) = \phi_1(h,0) = A_1 + B_1 h
\end{aligned}$$

giving $A_1 = 1, B_1 = -1/h, C_1 = -1/h$. This way, the basis functions get expressions

$$\phi_1(x,y) = 1 - x/h - y/h, \quad \phi_2(x,y) = x/h, \quad \phi_3(x,y) = y/h.$$

(ii) **2 p.** The corresponding gradients read as

$$\nabla\phi_1(x,y) = \begin{pmatrix} \partial\phi_1/\partial x \\ \partial\phi_1/\partial y \end{pmatrix} = \begin{pmatrix} -1/h \\ -1/h \end{pmatrix}, \quad \nabla\phi_2(x,y) = \begin{pmatrix} 1/h \\ 0 \end{pmatrix}, \quad \nabla\phi_3(x,y) = \begin{pmatrix} 0 \\ 1/h \end{pmatrix}.$$

For these constant values (together with the assumption of constant conductivity k_0), the stiffness coefficients of element e reduce to

$$k^e_{ij} = \int_e (k\nabla\phi^e_i)(x,y) \cdot \nabla\phi^e_j(x,y)\,\mathrm{d}x\mathrm{d}y = k_0 \nabla\phi^e_i \cdot \nabla\phi^e_j\,|e| = k_0\nabla\phi^e_i \cdot \nabla\phi^e_j\, h^2/2$$

where $|e|$ denotes the area of element e. Accordingly, the element stiffness matrix takes the form

$$\boldsymbol{K}^e = \begin{pmatrix} k^e_{11} & k^e_{12} & k^e_{13} \\ k^e_{21} & k^e_{22} & k^e_{23} \\ k^e_{31} & k^e_{32} & k^e_{33} \end{pmatrix} = \frac{k_0}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

(iii) **2 p.** The global stiffness matrix $\boldsymbol{K}$ of the system $\boldsymbol{Kd} = \boldsymbol{f}$ (and the force vector $\boldsymbol{f}$ correspondingly) is assembled such that each entry in the global stiffness matrix (corresponding to one corner node) gets contributions from the corresponding entries of the local 3×3 stiffness matrices $\boldsymbol{K}^e$ of elements sharing the node of the global entry.

If the (x,y)-origin lies in one corner of the domain occupied by the chosen element alone (as in this case), it holds that $k_{(0,0)} = k^e_{11}$. (Draw a picture for clarifying the situation!)

If the origin situates inside the domain and four elements surround it, for instance – say, e with corner points $(0,0),(h,0)$ and $(0,h)$, e' with corner points $(0,0),(0,-h)$ and $(h,0)$, e'' with corner points $(0,0),(-h,0)$ and $(0,-h)$ and e''' with corner points $(0,0),(0,h)$ and $(-h,0)$ – the corresponding entry in the global stiffness matrix is $k_{(0,0)} = k^e_{11} + k^{e'}_{11} + k^{e''}_{11} + k^{e'''}_{11}$. (Draw a picture for clarifying the situation!)

Problem 3

Let us consider a longish structure lying on supports at both ends. Let us model the static stress state of the structure by a simple beam bending model resulting in a one-dimensional boundary value problem: for given loadings $f = f(x)$, M_0 and M_L, find bending moment $M = M(x)$ such that

$$\begin{aligned} -M''(x) &= f(x) \quad \forall x \in (0, L), \\ M(0) &= M_0, \\ M(L) &= M_L, \end{aligned}$$

where L denotes the length of the beam and f stands for the distributed transversal loading, whereas moments M_0 and M_L are zeros for the simply supported beam in question.

(i) Derive the weak form of the problem setting.

(ii) Let us analyze the transversal (bending-caused) deflection of the same beam structure. Write down the corresponding boundary value problem in terms of deflection by following the classical Euler–Bernoulli beam model, and derive the associated weak form.

(iii) Form the finite element equation system for the displacement formulation of item (ii) by adopting one single element relying on Hermite basis functions

$$\begin{aligned} \phi_1(x) &= 1 - 3x^2 + 2x^3, \\ \phi_2(x) &= x - 2x^2 + x^3, \\ \phi_3(x) &= 3x^2 - 2x^3, \\ \phi_4(x) &= -x^2 + x^3. \end{aligned}$$

Model solutions for Problem 3

(i) **2 p.** The weak form can be derived by multiplying the differential equation by a test function $v = v(x)$, integrating over the interval and then integrating by parts once:

$$\begin{aligned} -M''(x)v(x) &= f(x)v(x), \quad x \in (0, L) \\ -\int_0^L M''(x)v(x)dx &= \int_0^L f(x)v(x)dx \\ [M'(x)v(x)]_0^L + \int_0^L M'(x)v'(x)dx &= \int_0^L f(x)v(x)dx. \end{aligned}$$

Since the boundary conditions of the strong form at $x = 0$ and $x = L$ set values (now assumed to be zeros) to the primary variable M, the test function must satisfy the corresponding conditions, i.e., $v(0) = 0 = v(L)$

which clean out the substitution term in the integral equation above. The weak form then reads as follows: Find $M = M(x)$ such that $M(0) = 0$, $M(L) = 0$ and

$$\int_0^L M'(x)v'(x)dx = \int_0^L f(x)v(x)dx$$

for all $v = v(x)$ satisfying $v(0) = 0$ and $v(L) = 0$.

(ii) **2 p.** From the bending moment problem above, the corresponding Euler–Bernoulli bending deflection problem can be derived by inserting the defition of bending moment $M(x) = -EIw''(x)$ into the problem setting: Find $w = w(x)$ such that

$$\begin{aligned}(EIw'')''(x) &= f(x), \quad x \in (0, L),\\ -EIw''(0) &= 0,\\ -EIw''(L) &= 0,\\ w(0) &= 0,\\ w(L) &= 0,\end{aligned}$$

where the last two conditions must be added as the problem is of order four and these conditions are the correct ones for a simply supported beam.

The weak form can be derived by multiplying the differential equation by a test function $v = v(x)$, integrating over the interval and by integrating by parts twice:

$$(EIw'')''(x)v(x) = f(x)v(x), \quad x \in (0, L)$$

$$\int_0^L (EIw'')''(x)v(x)dx = \int_0^L f(x)v(x)dx$$

$$[(EIw'')'(x)v(x)]_0^L - \int_0^L (EIw'')'(x)v'(x)dx = \int_0^L f(x)v(x)dx$$

$$[(EIw'')'(x)v(x)]_0^L - [EIw''(x)v'(x)]_0^L + \int_0^L (EIw'')(x)v''(x)dx = \int_0^L f(x)v(x)dx.$$

Inserting the natural boundary conditions $-EIw''(0) = M_0 = 0$ and $-EIw''(L) = M_L = 0$ into the second substitution term, and setting conditions $v(0) = 0, v(L) = 0$ in the first substitution term give the weak form: Find $w = w(x)$ such that $w(0) = 0$, $w(L) = 0$ and

$$\int_0^L (EIw'')(x)v''(x)dx = \int_0^L f(x)v(x)dx$$

for all $v = v(x)$ satisfying $v(0) = 0$ and $v(L) = 0$.

(iii) **2 p.** It should be noticed first that the given Hermite functions apply for interval $(0, L)$, i.e., for $L = 1$, whereas for a general interval x should be replaced by its dimensionless counterpart x/L.

Let us next check which basis functions correspond to the deflection degrees of freedom (deflection values at the end points): since

$$\begin{aligned} \phi_1(0) &= 1, \quad \phi_1(1) = 0 \\ \phi_2(0) &= 0, \quad \phi_2(1) = 0 \\ \phi_3(0) &= 0, \quad \phi_3(1) = 1 \\ \phi_4(0) &= 0, \quad \phi_4(1) = 0, \end{aligned}$$

functions ϕ_1 and ϕ_3 correspond to the deflection values at $x = 0$ and $x = 1$, respectively. Since the deflection distribution in the element (occupying the whole beam) is written as

$$w_h(x) = \phi_1(x)d_1 + \phi_2(x)d_2 + \phi_3(x)d_3 + \phi_4(x)d_4,$$

and this deflection must satisfy conditions $0 = w_h(0) = d_1$ and $0 = w_h(1) = d_3$, only two degrees of freedom and the corresponding two basis function survive to the final computations:

$$w_h(x) = \phi_2(x)d_2 + \phi_4(x)d_4.$$

Accordingly, the stiffness matrix of the problem reduces from 4×4 to 2×2:

$$\boldsymbol{K}^e = \begin{pmatrix} k_{22}^e & k_{24}^e \\ k_{42}^e & k_{44}^e \end{pmatrix}, \quad k_{ij}^e = \int_0^1 EI\phi_i''(x)\phi_j''(x)dx.$$

The corresponding force vector reads as

$$\boldsymbol{f}^e = \begin{pmatrix} f_2^e \\ f_4^e \end{pmatrix}, \quad f_i^e = \int_0^1 f(x)\phi_i(x)dx,$$

and the system of equations for finding $\boldsymbol{d}^e = (d_2, d_4)$ takes the form $\boldsymbol{K}^e\boldsymbol{d}^e = \boldsymbol{f}^e$.

With a constant EI, the system matrix would take the form

$$\boldsymbol{K}^e = \frac{EI}{L^3}\begin{pmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{pmatrix}$$

but solving the whole problem we should know the loading function.

CIV-E1060 Engineering Computation and Simulation
Examination, December 8, 2020 / Niiranen

This examination consists of 3 problems, each rated by the standard scale 1...6.

Problem 1

(i) From the definition of derivative for a one-variable scalar function $f = f(x)$ at point x defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

(1) derive the three basic numerical approximations for the derivative and (2) illustrate these concepts by a picture by using the polynomial function $f(x) = x(4-x)$ and its derivative at $x = 1$ as an example.

(ii) When analyzing an engineering problem with the finite element method, which are the possibilities available for an engineering consultancy office and its engineers to assure that the results obtained for the problem by a specific method provided by a chosen software are reliable? In your answer, briefly consider first (1) the possibilities originating from the nature and features of finite element methods and then (2) the possibilities valid for engineering modelling and computation in general.

(iii) When using the *Hermite* basis functions in the finite element method for *Euler–Bernoulli* beams, (1) which type of numerical integration scheme(s) should be used for evaluating the stiffness matrix entries and (2) why?

Model solutions to Problem 1

(i) **2 p**. (1) The forward finite difference follows the definition of derivative by simply dropping the limit away:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

The backward finite difference can be derived from the forward one by replacing h by $-h$, which gives

$$f'(x) \approx \frac{f(x-h) - f(x)}{-h} = \frac{f(x) - f(x-x)}{h}.$$

The central finite difference is the average of the forward and backward differences:

$$f'(x) \approx (\frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-x)}{h})/2 = \frac{f(x+h) - f(x-h)}{2h}.$$

(2) Drawing a picture with the curve $y = f(x) = x(4-x)$ tells a lot: the curve is a parabola which opens downwards and has zeros $x = 0$ and $x = 4$; the derivative is $f'(x) = -2x + 4$ and its value at $x = 1$ is $f'(1) = 1$ and it means the slope of the tangent of the curve at point $x = 1, f(1) = 3$.

The approximate derivatives give three approximations for the tangent: a line passing through points $(1, f(1))$ and $(1+h, f(1+h))$; a line passing through points $(1-h, f(1-h))$ and $(1, f(1))$; a line passing through points $(1-h, f(1-h))$ and $(1+h, f(1+h))$. The smaller h is, the closer the approximations are to the true tangent and each other. For drawing the curves, one can choose $h = 0.1$, for instance.

(For this example function, the central difference gives

$$f'(x) \approx \frac{(x+h)(4-(x+h)) - (x-h)(4-(x-h))}{2h} = ... = 4 - 2x$$

which is the exact value of the derivative, independently of h.)

(ii) **2 p.** (1) The nature and features of finite element methods enable the following: (a) increasing the number of elements step by step (meaning a set of finer meshes) and investigating if the results converge; (b) increasing the polynomial order for reaching higher accuracy; (c) trying a totally different method (element) provides a possibility for a double-check.

(2) Engineering modelling and computation in general support the following: (a) in structural mechanics, in particular, checking the balance between the external loadings and the reaction forces; (b) using another modelling option of a higher level (e.g., 2D instead of 1D) gives a validation reference; (c) another method or software can give a confirmation for the correctness of the results; (d) a comparison to a known solution (verification) of a simple model problem and a comparison to experimental results (validation) are sometimes available; (e) giving a task to two engineers (or two different groups) provides a possibility for a double-check. (It was not required to list all of these options.)

(iii) **2 p.** (1) The *Hermite* basis functions are qubic (third-order) polynomes, but the stiffness matrix entries of the finite element method for *Euler–Bernoulli* beams include an integral of the product of the second derivatives of two basis functions (and the bending rigidity which can be assumed to be constant):

$$k_{ij}^e = \int_e EI\varphi_i''(x)\varphi_j''(x)\,\mathrm{d}x.$$

Since the second derivatives are linear (first-order) polynomes, their product is a quadratic (second-order) polynome. In order to accurately integrate a quadratic polynome by a numerical quadrature, one can use the Gauss quadrature with $n = 2$ points due to the following reason:

(2) The Gauss quadrature with n points integrates accurately a polynome of order $2n - 1$. Since now the integrand is a polynome of order $m = 2$, the required number of points can be obtained by setting $2n - 1 = m = 2$ which gives $n = 3/2$ meaning that choosing $n = 2$ is enough (but $n = 1$ is not).

Problem 2

(i) Let us consider a diffusion–convection–reaction process modelled by using a one-dimensional model associated to the governing differential equation

$$-(ku')'(x) + cu'(x) + ru(x) = s(x), \quad x \in (0, L),$$

with a distributed source s and length L as well as the diffusivity (k), convection velocity (c) and reactivity (r) parameters. At one end of the problem domain (say, at $x = 0$), the primary problem variable is fixed to zero, whereas at the other end (say, at $x = L$) the flux of the primary variable is known – implying the boundary conditions of the problem.

(1) Derive the weak form of the problem. (2) What is the main difference this weak form implies to the corresponding finite element system equation when compared to the standard weak form of the diffusion problem?

(ii) Let us consider stationary heat conduction in a quadrangular domain (say, simply 1 m (or a m) wide and 2 m (or b m, with $b > a$) long). The physical problem can be modeled by relying on the first law of thermodynamics combined with the stationary state assumption, implying the partial differential equation

$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } \Omega$$

with the *Fourier* law building a constitutive relation between heat flux $\boldsymbol{q} = (q_1(x, y), q_2(x, y))$ and temperature $T = T(x, y)$ through thermal conductivity $k = k(x, y)$ written in the form

$$\boldsymbol{q} = -k\nabla T \quad \text{in } \Omega.$$

(1) Write down – the detailed derivation is not required – the weak form of the corresponding boundary value problem by assuming that along the boundary of the domain heat is fixed to a constant value.

(2) Find a rough approximation for the peak value of the temperature field, in the simplest case of constant conductivity and a constant heat source, by applying the finite element method of linear (first-order) triangular elements. You can use as many or as few elements as you consider reasonable for the given purpose.

Model solutions to Problem 2

(i) **2 p.** (1) The weak form can be derived by multiplying the differential equation by a test function $v = v(x)$ integrating over the interval and then integrating by parts once:

$$\begin{aligned}
&[-(ku')'(x) + cu'(x) + ru(x)]v(x) = s(x)v(x), \quad x \in (0, L),\\
&-(ku')'(x)v(x) + cu'(x)v(x) + ru(x)v(x) = s(x)v(x), \quad x \in (0, L),\\
&\int_0^L [-(ku')'(x)v(x) + cu'(x)v(x) + ru(x)v(x)]\,\mathrm{d}x = \int_0^L s(x)v(x)\,\mathrm{d}x,\\
&-\int_0^L (ku')'(x)v(x)\,\mathrm{d}x + \int_0^L cu'(x)v(x)\,\mathrm{d}x + \int_0^L ru(x)v(x))\,\mathrm{d}x\\
&\quad = \int_0^L s(x)v(x)\,\mathrm{d}x,\\
&-[(ku')(x)v(x)]_0^L + \int_0^L (ku')(x)v'(x)\,\mathrm{d}x + \int_0^L cu'(x)v(x)\,\mathrm{d}x + \int_0^L ru(x)v(x))\,\mathrm{d}x\\
&\quad = \int_0^L s(x)v(x)\,\mathrm{d}x,
\end{aligned}$$

Since the boundary condition at $x = 0$ is $u(0) = 0$, the test function must satisfy the corresponding condition $v(0) = 0$. Setting the boundary condition $-ku'(L) = q_L$ at $x = L$ finally implies the weak form: Find $u = u(x)$ such that $u(0) = 0$ and

$$\begin{aligned}
&\int_0^L (ku')(x)v'(x)\,\mathrm{d}x + \int_0^L cu'(x)v(x)\,\mathrm{d}x + \int_0^L ru(x)v(x))\,\mathrm{d}x\\
&\quad = \int_0^L s(x)v(x)\,\mathrm{d}x + ku'(L)v(L),
\end{aligned}$$

for all $v = v(x)$ satisfying $v(0) = 0$.

(2) Compared to the standard diffusion problem (terms corresponding to c and r are missing), this problem implies a nonsymmetric stiffness matrix since in the integrad of the c-term u does have a derivative but v does not. (The r-term is symmetric similarly to the k-term.)

(ii) **4 p.** (1) Since an essential boundary condition is given all around the boundary, the weak form is of the simplest possible form: Find $T = T(x, y)$ such that $T = T_0$ on the boundary Γ and

$$\int_\Omega (k\nabla T)(x, y) \cdot \nabla v(x, y)\,\mathrm{d}\Omega = \int_\Omega f(x, y)\, v(x, y)\,\mathrm{d}\Omega$$

for all $v = v(x, y)$ satisfying $v = 0$ on Γ.

(2) For finding a rough finite element approximation for the exact solution (which in this problem is a bubble function growing from the constant value of the boundaries to a peak value in the middle), let us first utilize the symmetry of the problem by considering only one quadrant of the rectangle: if the rectangle is placed such that the 2 m long sides form the bottom and top edges, we consider the quadrant rectangle in the upper left corner. Accordingly, let us fix the origin of an xy-coordinate system to the middle point of the left 1 m long edge of the original rectangle. Hence, the quadrant in consideration is described in this coordinate system by the domain $(0, 1) \times (0, 1/2)$. This domain is divided into two triangles: the first having the vertices $(0, 0), (1, 0), (0, 1/2)$; the second having the vertices $(1, 0), (1, 1/2), (0, 1/2)$. Three of these vertices situate on the boundaries of the original rectangle which means that at those points $T = T_0$, and only the vertex $(1, 0)$ (shared by both triangles) lies inside the original rectangle (in the center where the peak temperature should appear).

Let us consider the first triangle having the nodes $(0, 0), (1, 0), (0, 1/2)$ and the corresponding linear shape functions

$$\begin{aligned}
N_1(x, y) &= 1 - x/L_1 - y/L_2 = 1 - x - 2y,\\
N_2(x, y) &= x/L_1 = x,\\
N_3(x, y) &= y/L_2 = 2y,
\end{aligned}$$

with the base and height dimensions $L_1 = 1$, $L_2 = 1/2$ (this way the shape functions satisfy the required conditions: decrease from value 1 linearly to 0). In the final system equation, for this triangle the only essential stiffness entry is the one corresponding to the node $(1, 0)$ (other nodes lie on the boundary) giving

$$k^1_{22} = \int_e (k\nabla N_2)(x, y) \cdot \nabla N_2(x, y)\,\mathrm{d}\Omega = \int_e k(1, 0) \cdot (1, 0)\,\mathrm{d}\Omega = kL_1L_2/2 = k/4.$$

The corresponding load component is

$$f^1_2 = \int_e f_0 N_2(x, y)\,\mathrm{d}\Omega = f_0/2.$$

For considering the second triangle having the nodes $(1, 0), (1, 1/2), (0, 1/2)$, we move the origin of the coordinate system to the point $(1, 1/2)$ and direct the y-axis downwards. In the final system equation, for this triangle the only essential stiffness entry is (in this new system) the one corresponding to the node $(0, 1/2)$ giving

$$k^2_{33} = \int_e (k\nabla N_3)(x, y) \cdot \nabla N_3(x, y)\,\mathrm{d}\Omega = \int_e k(0, 2) \cdot (0, 2)\mathrm{d}\Omega = 2kL_1L_2 = k.$$

The corresponding load component is

$$f^2_3 = \int_e f_0 N_3(x, y)\mathrm{d}\Omega = f_0/4.$$

Both triangles contribute to the nonzero degree of freedom $T_m = T^1_{22} = T^2_{33}$ (situating in the middle of the original rectangle) and hence form the finite element system equation

$$(5k/4)T_m = (k^1_{22} + k^2_{33})T_m = f^1_2 + f^2_3 = 3f_0/4$$

which gives the peak value approximation $T_m = 3f_0/5k$.

Problem 3

(i) Briefly explain what are the key reasons for using the so-called reference element approach in finite element methods.

(ii) Give one mathematical and one physical reason for the use of *Hermite* basis functions for the *Euler–Bernoulli* beam bending problem instead of the *Lagrange* basis functions considered as the standard ones for finite element methods.

(iii) Let us use the *Hermite* basis functions for the following *Euler–Bernoulli* beam bending problem:

- the distributed loading is constant;
- the bending rigidity is constant;
- one end of the beam (say, at $x = 0$) is clamped (say, built into a wall as for a cantilever);
- the other end (say, at $x = L$) is free to move in the direction perpendicular to the central axis of the beam, but the rotation at this end point has been prevented by a roller support rigidly joined to beam but sliding along a rail crossing the end point in the direction perpendicular to the central axis.

Construct the corresponding finite element equation system for the problem by adopting one single element relying either on the actual line segment $(0, L)$ or the typical reference element line segment $(-1, 1)$ – for both of which, the *Hermite* basis functions can be found in the course material – or by relying on the line segment $(0, 1)$ for which the *Hermite* basis functions take the following form:

$$\begin{aligned}
\phi_1(x) &= 1 - 3x^2 + 2x^3,\\
\phi_2(x) &= x - 2x^2 + x^3,\\
\phi_3(x) &= 3x^2 - 2x^3,\\
\phi_4(x) &= -x^2 + x^3.
\end{aligned}$$

Model solutions to Problem 3

(i) **1 p.** The FEM requires calculating stiffness matrix and force vector entries in every element of the mesh, and the reference element enables doing this in an efficient way: defining and dealing with shape functions and numerical integration cab be accomplished only in one simple element (although some linear transformations and changes of variables are needed) and the results are then repeated when forming the system equations in the assemly proccess of the system equation.

(ii) **2 p.** (1) A mathematical reason: The strain energy in the variational (weak) formulation includes second-order derivatives of the primary problem variable (deflection), which means that for obtaining a conforming finite element method the trial and test functions need to be C^1-continuous functions (that the *Hermite* basis functions provide). (The *Lagrange* basis functions provide only C^0-continuous approximation functions.)

(2) A physical reason: It is natural to give external moments for beams as boundary conditions, and moments should be associated to rotations (moment times rotation gives energy, which means that they are conjugate quantities of each other). The *Hermite* basis functions provides at the end points of each element both deflection and rotation degrees of freedom, hence serving both external forces and moments, respectively. (The *Lagrange* basis functions provide only deflection degrees of freedom conjugate to forces.)

(iii) **3 p.** One single *Hermite* element has deflection and rotation degrees of freedom at both ends. Since one end of the beam (say, at $x = 0$) is clamped both degrees of freedom at this node (d_1, d_2 below) are set to zero, and since the rotation of the other end (say, at $x = L$) is prevented the rotation degree of freedom at this node (d_4 below) is set to zero – leaving only one deflection degree of freedom (d_3 below) as the only unknown of the problem:

$$w_h(x) = N_1(x)d_1 + ... + N_4(x)d_4 = N_3(x)d_3$$

The basis function corresponding to this deflection degree of freedom is

$$N_3(x) = 3(x/L)^2 - 2(x/L)^3$$

since $N_3(L) = 1$ and $N_3'(L) = 0$ (one can assume, for simplicity, that $L = 1$). The corresponding stiffness matrix entry is

$$k_{33} = \int_0^L EIN_3''(x)N_3''(x)\,\mathrm{d}x = ... = 12EI/L^3.$$

The corresponding load vector entry is

$$f_3 = \int_0^L f_0 N_3(x)\mathrm{d}x = ... = -f_0 L/2.$$

The finite element system equation takes the form

$$(12EI/L^3)d_3 = k_{33}d_3 = f_3 = -f_0 L/2$$

which gives the deflection value approximation $w_h(L) = d_3 = -f_0 L^4/24EI$.