MS-E1652 Computational methods for differential equations

Course exam and Exam, 9:00–13:00, December 13, 2021 **Hyvönen/Hirvi**

Access to extra materials and resources is not restricted, but consulting other people is strictly forbidden. All solutions must indicate how the end results and conclusions were reached via calculations by hand, with a credible number of intermediate stages presented.

The grade based on the combination of the course exam and the exercise points is deduced by accounting for the four solutions that produce the most points; if a student, e.g., gets 6, 2, 4, 4 *and* 5 *points from the individual exam problems, then the number of exam points considered in connection to the exercise points is* $19 = 6+4+4+5$. *The grade based on the mere final exam is deduced by accounting for all five solutions. The better of the two alternatives will correspond to the final grade.*

1. Consider the *linear multistep method* (LMM)

$$x_{j+3} - x_j = \frac{3}{2}h\big(f_{j+3} + f_j\big) \qquad j = 0, 1, 2, \dots \; . \tag{1}$$

Consider the following questions/tasks. Justify your answers.

(a) Is (1) an explicit or an implicit LMM?

(b) Prove that (1) is consistent of order $p = 2$, but not of order $p = 3$.

(c) Is (1) zero-stable?

(d) Why is zero-stability an indispensable property of any functional LMM?

2. Determine the region of absolute stability $\mathcal{R} \subset \mathbb{C}$ for the LMM (1).

3. Consider the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{1}{3}h\big(2k_1 + k_2\big), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + \frac{3}{4}h, x_j + \frac{3}{4}hk_1\Big) \end{aligned} \tag{2}$$

(a) Is (2) an explicit or an implicit RK method? Why?

(b) What is the stability function corresponding to (2)?

(c) Determine the region of absolute stability $\mathcal{R} \subset \mathbb{C}$ for (2)? (Hint: it is a disk.)

(d) Based on the stability function, what would you expect to be the consistency order of (2)? Justify your answer.

4. When the initial/boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in (0,1), \end{cases}$$

is spatially discretized by the standard central second order difference approximation, one ends up with the following initial value problem:

$$U'(t) = A\,U(t), \qquad U(0) = G, \tag{3}$$

for all $t \geq 0$. Here, $G = [g(x_1), \dots, g(x_m)]^{\mathrm{T}}$ and $U(t) \approx [u(x_1,t), \dots, u(x_m,t)]^{\mathrm{T}}$, with $x_j = jh$ and $h = 1/(m+1)$ being the mesh parameter.

(a) Write down the matrix $A \in \mathbb{R}^{m \times m}$. (It is enough describe the structure of A — you need not present an actual proof.)

(b) Introduce Heun's method for numerically solving (3). Let $\delta > 0$ be the time step size and denote by U_k the approximation of $U(k\delta)$ for $k = 0, 1, 2, \ldots$.

(c) For which $\delta > 0$ is the introduced method (for sure) absolutely stable, that is,

$$\lim_{k \to \infty} U_k = 0 \in \mathbb{R}^m$$

for any $G \in \mathbb{R}^m$? Justify your answer.

(d) Let $T > 0$ be fixed. What is the largest value of $p \in \mathbb{N}$ for which the estimate

$$\big| U_k - U(k\delta) \big| \leq C\delta^p \qquad \text{for all } G \in \mathbb{R}^m$$

holds for any $k \in \mathbb{N}$ such that $k\delta \in (0, T]$ and some $C > 0$ that may depend on m and T. Is it possible to choose C to be independent of m? (You need not deduce the order of Heun's method, but you can refer, e.g., to the lecture notes for that information.)

5. Consider the initial value problem

$$u'(t) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} u(t), \qquad u(0) = u_0 \in \mathbb{R}^2. \tag{4}$$

(a) Prove that

$$\lim_{t \to \infty} |u(t)| = \infty$$

if $u_0 \neq 0$.

(b) Apply the implicit Euler's method to (4). For which values of the time step $h > 0$ (with $h \neq 0.5$ and $h \neq 1$) does the produced numerical solution satisfy

$$\lim_{j \to \infty} |u_j| = \infty$$

for all $\mathbb{R}^2 \ni u_0 \neq 0$? (You can assume infinite computational precision.)