Aalto University School of Science
Department of Mathematics and Systems Analysis Korte

MS-C1350 Partial differential equations, fall 2021

Course exam on 20 Dec 2021 at 9:00-13:00

This set of problems is for the participants of the course in the fall 2021 and affects 40% in the grading of the course.

Remember to explain carefully your answers. Remember to answer to Problem 1 (multiple choice question) as well.

Each of the problems 2–5 is worth 6 points.

2. Suppose that u is a smooth solution to the heat equation $u_t = \Delta u$.
   (a) Show that $v(x,t) = u(\lambda t, \lambda^2 t)$ is a solution to the heat equation for every $\lambda \in \mathbb{R}$.
   (b) Show that $w(x,t) = x \cdot \nabla u(x,t) + 2tu_t(x,t)$ is also a solution to the heat equation.

3. Let $f \in C^\infty(\mathbb{R}^n)$. Consider solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to the problem
$$\begin{cases} \Delta u = f(x) & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0 & x \in \partial\Omega. \end{cases}$$
   Prove that the problem cannot have exactly one solution. Which condition does f have to satisfy if the problem has solutions?

   Hint: Green's identities

4. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and $f \in C^2(\mathbb{R}^2)$. Let us consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ in Ω with boundary values $u(x,0) = u(x,1) = 0$, when $0 < x < 1$, and $u(0,y) = 0$, $u(1,y) = f(y)$, when $0 < y < 1$.
   (a) Reduce the problem to two ODEs by using separation of variables.
   (b) Solve the separated equations to find special solutions.
   (c) Let $f(t) = \sin(5\pi t)$. Find the explicit solution u.

5. Consider the following one-dimensional non-homogeneous problem for the wave equation:
$$\begin{cases} u_{tt} - u_{xx} = xt, & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$
   Find a solution to this problem using Duhamel's principle and d'Alembert's formula.